Trajectory and smooth attractors for Cahn-Hilliard equations with inertial term

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Abstract

The paper is devoted to a modification of the classical Cahn-Hilliard equation proposed by some physicists. This modification is obtained by adding the second time derivative of the order parameter multiplied by an inertial coefficient $\varepsilon > 0$ which is usually small in comparison to the other physical constants. The main feature of this equation is the fact that even a globally bounded nonlinearity is "supercritical" in the case of two and three space dimensions. Thus the standard methods used for studying semilinear hyperbolic equations are not very effective in the present case. Nevertheless, we have recently proven the global existence and dissipativity of strong solutions in the 2D case (with a cubic controlled growth nonlinearity) and for the 3D case with small ε and arbitrary growth rate of the nonlinearity (see [26, 25]). The present contribution studies the long-time behavior of rather weak (energy) solutions of that equation and it is a natural complement of the results of our previous papers [26] and [25]. Namely, we prove here that the attractors for energy and strong solutions coincide for both the cases mentioned above. Thus, the energy solutions are asymptotically smooth. In addition, we show that the non-smooth part of any energy solution decays exponentially in time and deduce that the (smooth) exponential attractor for the strong solutions constructed previously is simultaneously the exponential attractor for the energy solutions as well. It is worth noting that the uniqueness of energy solutions in the 3D case is not known yet, so we have to use the so-called trajectory approach which does not require the uniqueness. Finally, we apply the obtained exponential regularization of the energy solutions for verifying the dissipativity of solutions of the 2D modified Cahn-Hilliard equation in the intermediate phase space of weak solutions (in between energy and strong solutions) without any restriction on ε.

Key words: trajectory attractors, smooth global attractors, singularly perturbed Cahn-Hilliard equation.

AMS (MOS) subject classification: 35B40, 35B41, 82C26.

1 Introduction

In a series of contributions, P. Galenko *et al.* (see [17, 18, 19, 20]) have proposed to modify the celebrated Cahn-Hilliard equation (see [10], cf. also the review [31]) in order to account for nonequilibrium

effects in spinodal decomposition (cf. [9], see also [24, 29]). The basic form of this modification reads

$$\varepsilon u_{tt} + u_t - \Delta(-\Delta u + f(u)) = g, \tag{1.1}$$

on $\Omega \times (0, +\infty)$, Ω being a bounded smooth subset of $\mathbb{R}^{\mathrm{N}}$, $\mathrm{N} \leq 3$. Here $\varepsilon \in (0, 1]$, f is the derivative of a nonconvex potential (e.g., $f(r) = r(r^2 - 1)$) and g is a given (time-independent) function.

The longtime behavior of equation (1.1) already drew the attention of mathematicians (see the pioneering [14], cf. also the more recent [7, 21, 23, 35, 36]). However, all these contributions were essentially devoted to the one-dimensional case which can now be considered well known. There also have been further works devoted to higher dimensions (see [22, 27], cf. also [13, 34] for memory effects) but they are all characterized by the presence of viscosity terms which imply the instantaneous regularization of solutions. This is not the case of (1.1).

The mere existence of energy bounded solutions (see Def. 2.1 for this terminology) was proven in [33] for $N = 3$. We recall that this work was mainly devoted to the longtime behavior of such solutions based on the multi-valued semigroup approach to the problems without uniqueness developed by A.V. Babin and M.I. Vishik [3] (see also [2], [5, 6] and references therein). The existence result was then generalized to a nonisothermal system with memory in [28]. Nevertheless, the existence of energy bounded solutions is not a delicate issue and can be carried out by means of a standard Galerkin procedure. In addition, both the quoted results were proven supposing f of cubic controlled growth, but the existence also holds when f has a generic polynomial growth (cf. Thm. 2.2 below). On the contrary, uniqueness of such solutions is much harder to prove. This was eventually done in [26] for $\mathrm{N} = 2$, assuming f of cubic controlled growth, along with a number of other results (e.g., existence of smoother solutions, global attractors and exponential attractors). Uniqueness of energy bounded solutions is still open in the case $\mathrm{N} = 3$ (and also when $\mathrm{N} = 2$ for supercubic f). More recently, an extension to a version with memory has been studied in [12]. However, the existence of stronger solutions was shown in [25] for ε small enough. This fact enabled the authors to construct a dynamical system acting on a suitable phase space (depending on ε) and to prove the existence of the global attractor as well as of an exponential attractor.

Speaking of global attractors in the cases $\mathrm{N} = 2$ and $\mathrm{N} = 3$, the results obtained in [26] and [25] are not fully satisfactory. Indeed, in the former case we proved the existence of the global attractors both for energy bounded solutions and for "quasi-strong" solutions (see Def. 2.1 again), but we could not say whether they coincide. In the case $\mathrm{N} = 3$ we only established the existence of the global attractor for quasi-strong solutions, while the unique available result on global attractors for energy bounded solutions was in [33]. Inspired by [37], here we intend to bridge this gap.

First, in Sections 2 and 3, we construct the proper attractor for the energy solutions of (1.1). Since we do not have the uniqueness for the energy solutions (in the 3D case as well as in the 2D case with the super-cubic growth rate), we use the so-called trajectory dynamical system approach developed by V.V. Chepyzhov and M.I. Vishik (see [11]) and construct the so-called trajectory attractor associated with energy solutions of problem (1.1). However, the class of all energy solutions which satisfy the weakened form of energy inequality used in [11] in their construction of trajectory attractors for damped hyperbolic equations is too large for our purposes and we restrict ourselves to consider only the energy solutions which can be obtained by Galerkin approximations (analogously to [37], see also [33]). Note that, although the trajectory attractor constructed here is very close (and even formally equivalent) to the generalized attractor obtained in [33] via the multi-valued approach, it is much more convenient for our further investigation.

Then, in Section 4, we establish that each complete bounded solution belonging to the trajectory attractor is a strong solution to (1.1) at least for all times smaller than a time sufficiently close to $-\infty$. This kind of backward smoothness was firstly obtained in [37] for damped wave equations with supercritical nonlinearities. Here, the proof is however based on partly different and more simple arguments (see Thm. 4.1).

The backward smoothness is the basic ingredient which allows us to show (in Section 5):

(i) if $\mathrm{N} = 2$ (and f has cubic controlled growth) the global attractor for energy bounded solutions coincides with the one for quasi-strong solutions (and, in particular, it is smooth);

(ii) if $\mathrm{N} = 3$ the trajectory attractor consists of complete bounded strong solutions if ε is small enough.

Thus, in both cases, we have the asymptotic regularization of energy solutions.

In Section 6, in both cases mentioned above, we establish that every energy solution regularizes *exponentially* as $t \to +\infty$, i.e., such solutions can be split into the sum of two functions, one of which is smooth and bounded and the other tends to zero exponentially as time tends to infinity. This result seems new even for the well-known damped semi-linear wave equation with supercritical nonlinearity (a similar property has been shown in [37] *only* under the additional assumptions that all equilibria are hyperbolic). In addition, we present an alternative approach to demonstrate the exponential asymptotic regularization when $\varepsilon > 0$ is small enough. This method does not use the backward regularization or exploit the global Lyapunov functional and can be therefore applied, e.g., to non-autonomous equations or to the case of unbounded domains.

Finally, in Section 7, taking advantage of the exponential regularization and the transitivity of exponential attraction, we prove that, again in both the above cases, the energy solutions approach *exponentially* fast the exponential attractor for strong solutions which has been constructed in [25] and [26] . Thus, this strong exponential attractor is the exponential attractor for the energy solutions as well. In addition, we use the obtained exponential regularization to solve one problem for the 2D case which remained open in [26], namely, the dissipativity in the intermediate phase space of weak solutions (between the energy and strong solutions) with no restrictions on ε.

To conclude the introduction, we note that, although we endow equation (1.1) with the boundary and initial conditions

$$u(t) = \Delta u(t) = 0, \quad \text{ on } \partial\Omega,\ t > 0, \tag{1.2}$$

$$u(0) = u_0, \quad u_t(0) = u_1, \quad \text{ in } \Omega, \tag{1.3}$$

other boundary conditions, like no-flux or periodic, could also be handled (see [7, 14, 21]) with only technical modifications.

2 Functional setup and existence of solutions

Let us set $H := L^2(\Omega)$ and denote by $(\cdot,\cdot)$ the scalar product both in H and in $H \times H$, and by $\|\cdot\|$ the related norm. The symbol $\|\cdot\|_X$ will indicate the norm in the generic (real) Banach space X. Next, we set $V := H_0^1(\Omega)$, so that $V' = H^{-1}(\Omega)$ is the topological dual of V. The duality between V' and V will be noted by $\langle\cdot,\cdot\rangle$. The space V is endowed with the scalar product

$$((v, z)) := \int_\Omega \nabla v \cdot \nabla z, \quad \forall\, v, z \in V, \tag{2.1}$$

and the corresponding induced norm. We shall denote by A the Riesz operator on V associated with the norm above, namely,

$$A : V \to V', \qquad \langle Av, z\rangle = ((v, z)) = \int_\Omega \nabla v \cdot \nabla z, \quad \forall\, v, z \in V. \tag{2.2}$$

Abusing notation slightly, we shall also indicate by the same letter A the restriction of the operator defined in (2.2) to the set $D(A) = H^2(\Omega) \cap V$, i.e., the unbounded operator defined as

$$A = -\Delta \quad \text{ with domain } D(A) = H^2(\Omega) \cap V \subset L^2(\Omega). \tag{2.3}$$

Starting from A one can define the family of Hilbert spaces

$$H^{2s} = D(A^s), \quad s \in \mathbb{R},$$

with scalar product $(A^s\cdot, A^s\cdot)$. It is well known that $H^{s_1} \subset H^{s_2}$ with dense and compact immersion when $s_1 > s_2$. Then, we introduce the scale of Hilbert spaces

$$\mathcal{V}_s^\varepsilon := D(A^{\frac{s+1}{2}}) \times \sqrt{\varepsilon} D(A^{\frac{s-1}{2}}), \tag{2.4}$$

so that we have, in particular, $\mathcal{V}_0^\varepsilon = V \times V'$ and, analogously, $\mathcal{V}_1^\varepsilon = (H^2(\Omega) \cap V) \times H$. The spaces $\mathcal{V}_s^\varepsilon$ are naturally endowed with the graph norm

$$\|(u, v)\|^2_{\mathcal{V}_s^\varepsilon} := \|A^{\frac{s+1}{2}} u\|_H^2 + \varepsilon \|A^{\frac{s-1}{2}} v\|_H^2. \tag{2.5}$$

Regarding the nonlinear function f, we assume that $f \in C^3(\mathbb{R};\mathbb{R})$ with $f(0) = 0$ satisfies, for some $p \in [0, \infty)$,

$$\liminf_{|r|\nearrow+\infty} \frac{f(r)}{r} > -\lambda_1; \tag{2.6}$$

$$\exists\, \lambda \in [0, +\infty) \text{ and } \delta \in [0, \infty): \quad f'(r) \geq -\lambda + \delta |r|^{p+2}, \quad \forall\, r \in \mathbb{R}; \tag{2.7}$$

$$\exists\, M \geq 0: \quad |f'''(r)| \leq M(1 + |r|^p), \quad \forall\, r \in \mathbb{R}. \tag{2.8}$$

Here $\lambda_1 > 0$ is the first eigenvalue of A. Note that f can be a polynomial of arbitrarily large odd degree with positive leading coefficient. If we indicate as F the potential of f (i.e., a suitable primitive of f), we can always suppose that

$$F(r) \geq -\frac{\kappa}{2} r^2, \tag{2.9}$$

for some $\kappa < \lambda_1$. By (2.7), we also have that F is λ-convex. When $p > 2$ we will need to suppose $\delta > 0$ in (2.7).

Finally, we let

$$g \in H. \tag{2.10}$$

System (1.1)-(1.3) can then be reformulated as

Problem P_ε. Find a pair (u, u_t) satisfying

$$\varepsilon u_{tt} + u_t + A(Au + f(u)) = g, \tag{2.11}$$

$$u|_{t=0} = u_0, \quad u_t|_{t=0} = u_1. \tag{2.12}$$

Here, it is intended that the first two relations hold at least for almost any time t in the life span of the solution. Actually, we will consider in the sequel both local and global in time solutions. In the sequel we will frequently write U for the couple (u, u_t) and U_0 for (u_0, u_1) (the same convention will be kept for other letters, e.g., we will write $V = (v, v_t)$). Moreover, for the sake of brevity, solutions will be sometimes noted simply as u, or as U, rather than as (u, u_t).

Speaking of regularity, we can now introduce the *energy* associated with (1.1) as

$$\mathcal{E}_\varepsilon : \mathcal{V}_0^\varepsilon \to \mathbb{R}, \qquad \mathcal{E}_\varepsilon(u, v) := \frac{1}{2} \|(u, v)\|_0^2 + \int_\Omega F(u) - \langle g, A^{-1} u \rangle. \tag{2.13}$$

Assumptions (2.8) and (2.10) suffice to guarantee that $\mathcal{E}_\varepsilon$ is finite for all $(u, v) \in \mathcal{V}_1^\varepsilon$. However, if $p > 2$ then $\delta > 0$ in (2.7) is needed, if $(u, v) \in \mathcal{V}_0^\varepsilon$. For this reason we introduce the function space

$$\mathcal{X}_0^\varepsilon := \big\{(u, v) \in \mathcal{V}_0^\varepsilon : u \in L^{p+4}(\Omega)\big\}, \tag{2.14}$$

which is endowed with the graph metrics. For instance, with some abuse of language, we will write

$$\|(u, v)\|_{\mathcal{X}_0^\varepsilon}^2 := \|(u, v)\|_{\mathcal{V}_0^\varepsilon}^2 + \|u\|_{L^{p+4}(\Omega)}^{p+4}. \tag{2.15}$$

It is then clear from (2.8) that $\mathcal{E}_\varepsilon$ is locally finite in $\mathcal{X}_0^\varepsilon$. Of course, thanks to (2.6), $\mathcal{E}_\varepsilon$ is in any case bounded from below on the whole $\mathcal{V}_0^\varepsilon$. The above discussion leads to the following definition (see [25, 26]).

Definition 2.1. *We say that a solution to P_ε defined on some time interval $(0, T)$ is an* energy bounded solution, *or, more concisely,* energy solution, *if $(u, u_t) \in L^\infty(0, T; \mathcal{V}_0^\varepsilon)$ and $\mathcal{E}_\varepsilon(u, u_t) \in L^\infty(0, T)$. If $(u, u_t) \in L^\infty(0, T; \mathcal{V}_1^\varepsilon)$, we say instead that u is a* weak solution. *If $(u, u_t) \in L^\infty(0, T; \mathcal{V}_2^\varepsilon)$, u is named* quasi-strong solution, *while u is a* strong solution *if $(u, u_t) \in L^\infty(0, T; \mathcal{V}_3^\varepsilon)$.*

Thus, for energy solutions, (2.11) has to be interpreted as an equation in $D(A^{-2})$ in the case when $p > 2$ (and hence $\delta > 0$). Indeed, in this case $f(u(t)) \in L^{\frac{p+4}{p+3}}(\Omega) \subset D(A^{-1})$ for almost any $t \in (0, T)$. If $p \leq 2$, then of course we can say more: it is now $f(u(t)) \in L^{6/5}(\Omega) \subset V'$ and (2.11) holds in $D(A^{-3/2})$. In any case, we can say that any energy solution $U = (u, u_t)$ lies in $L^\infty(0, T; \mathcal{X}_0^\varepsilon)$. Passing to weak solutions, then (2.11) holds in $D(A^{-1})$ since it is now $f(u(t)) \in H$, thanks to the

embedding $H^2(\Omega) \hookrightarrow C(\overline{\Omega})$. For the same reason, (2.11) can be interpreted as a V'-equation for quasi-strong solutions, and, of course, for strong solutions it holds almost everywhere in $\Omega \times (0,T)$. Observe that, despite of the name, energy bounded solutions are *weaker* than weak solutions.

We also notice that a comparison in (2.11) gives that $u_{tt} \in L^\infty(0,T,H)$ for strong solutions, $u_{tt} \in L^\infty(0,T,V')$ for quasi-strong solutions, $u_{tt} \in L^\infty(0,T,D(A^{-1}))$ for weak solutions, and $u_{tt} \in L^\infty(0,T,D(A^{-2}))$ for energy solutions. This immediately leads to $U = (u,u_t) \in C^0_w([0,T];\mathcal{V}^\varepsilon_i)$ with, respectively, $i = 3,2,1,0$, where $C^0_w([0,T];X)$ is defined as (X being a real Banach space)

$$C^0_w([0,T];X) := \left\{ v \in L^\infty(0,T;X) : \ \langle \phi, v(\cdot) \rangle \ \in C^0([0,T]), \ \ \forall \phi \in X' \right\}.$$

Therefore solutions can be evaluated pointwise in time and initial conditions (2.12) in $\mathcal{V}^\varepsilon_i$, $i = 3,2,1,0$, have a well-defined meaning in all the cases.

We conclude the section by stating the existence theorem

Theorem 2.2. *Let the assumptions* (2.6)-(2.8) *and* (2.10) *hold, and let*

$$(u_0, u_1) \in \mathcal{X}_0. \tag{2.16}$$

Then, if either $p \le 2$ *in* (2.7)-(2.8) *or* $\delta > 0$ *in* (2.7), *there exists at least one* global in time *energy solution to Problem* P_ε, *which additionally satisfies the following dissipation inequality*

$$\|U(t)\|^2_{\mathcal{X}^\varepsilon_0} + \int_t^{+\infty} \|u_t(s)\|^2_{V'} \, \mathrm{d}s \le C \|U_0\|^2_{\mathcal{X}^\varepsilon_0} e^{-\kappa t} + C(1 + \|g\|^2), \quad \forall t \ge \tau \ge 0, \tag{2.17}$$

where the positive constants κ *and* C *are independent of* ε.

From now on we let the assumptions of Theorem 2.2 hold, unless otherwise specified. Moreover, we will not stress the dependence on ε till the final section.

Although the proof of Theorem 2.2 is standard, we report here below some highlights for the reader's convenience.

Proof. The proof is essentially based on the Faedo-Galerkin scheme described in the next section and on a couple of estimates which, for simplicity, are performed here by working directly (albeit formally) on the original problem rather than on its approximation. Firstly, testing (2.11) by $A^{-1}u_t$, we easily derive the *energy equality*

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}_\varepsilon(U) + \|u_t\|^2_{V'} = 0 \tag{2.18}$$

(in fact, the above is a true equality just at the regularized (Galerkin) level, but will turn into an inequality when taking the limit). Second, multiplying (2.11) by $A^{-1}u$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \Big[\varepsilon \langle u_t, A^{-1}u \rangle + \frac{1}{2} \|u\|^2_{V'} \Big] - \varepsilon \|u_t\|^2_{V'} + \|u\|^2_V + (f(u), u) - (g, A^{-1}u) = 0. \tag{2.19}$$

We then notice that a combination of assumptions (2.6)-(2.7) gives

$$\|u\|^2_V + (f(u), u) \ge \kappa_1 \|u\|^2_V + \kappa_2 \int_\Omega F(u) - c \ge \kappa_3 \|u\|^2_V - c, \tag{2.20}$$

for suitable positive constants κ_i, $i = 1,2,3$, only depending on λ_1 and λ. In particular, thanks to (2.6), κ_2 can be chosen so small that the latter inequality holds even in case $\delta = 0$ in (2.7).

Using (2.20), it is a standard matter to verify that, multiplying (2.19) by a (suitably small) constant $\alpha > 0$, and adding the result to (2.18), gives, for some $\kappa > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{Y}_\varepsilon + \kappa \mathcal{Y}_\varepsilon \le c, \tag{2.21}$$

where we noted as $\mathcal{Y}_\varepsilon$ the functional obtained by taking $\mathcal{E}_\varepsilon$ and summing to it α times the terms in square brackets in (2.19) and adding also a further quantity of the form $C_*(1 + \|g\|^2)$, where $C_* > 0$

is a suitable constant. Actually, by (2.15), it is easy to check that, if C_* is taken large enough, then for some positive κ_i, $i = 4, \dots, 7$, and $c > 0$ there holds

$$\kappa_4 \|U\|^2_{\mathcal{V}_0^\varepsilon} + \delta \|u\|^{p+4}_{L^{p+4}(\Omega)} \le \kappa_5 \mathcal{E}_\varepsilon \le \mathcal{Y}_\varepsilon \le \kappa_6 \mathcal{E}_\varepsilon + c(1 + \|g\|^2_H) \le \kappa_7 \|U\|^2_{\mathcal{X}_0^\varepsilon} + c(1 + \|g\|^2_H). \tag{2.22}$$

Thus, integrating (2.21) over $(0, t)$ and using (2.22), we obtain an inequality analogue to (2.17), but without the integral term on the left hand side. To get a control of it, it is however sufficient to go back to (2.18), integrate it over $(t, +\infty)$, and refer once more to (2.22).

Once one has obtained (2.17) at the approximated level, it is then a standard procedure to pass to the limit. For instance, assumptions (2.7)-(2.8) and the Aubin-Lions compactness lemma entail weak star convergence of $f(u)$ to the right limit, holding in the space $L^\infty(0, T; L^{6/5}(\Omega))$ if $p \le 2$, and in $L^\infty(0, T; L^{\frac{p+4}{p+3}}(\Omega))$ if $p > 2$ (and hence $\delta > 0$). Then, also (2.17) passes to the limit inferior by standard semicontinuity arguments. The proof is complete. ∎

Remark 2.3. It is not difficult to realize that Theorem 2.2 still holds when $f \in C^0(\mathbb{R}; \mathbb{R})$ with $f(0) = 0$ satisfies, in place of (2.6)-(2.8) and for some $p \in (0, \infty)$,

$$\exists\, M_0 \ge 0 : \quad |f(r)| \le M_0(1 + |r|^{p+3}), \quad \forall\, r \in \mathbb{R}$$
$$\exists\, \lambda_0 \in [0, +\infty) \text{ and } \delta_0 \in (0, \infty) : \quad f(r)r \ge -\lambda_0 + \delta_0 |r|^{p+4}, \quad \forall\, r \in \mathbb{R}.$$

If $p \le 2$, the last condition can be replaced by (2.6). These assumptions on f also suffice to establish the existence of the trajectory attractor (see Theorem 3.4 and related corollaries in the next section).

3 The trajectory dynamical system

The existence of a global energy solution (see Theorem 2.2) can be proven by means of a Faedo-Galerkin procedure similar to the one used in [37] for the damped semilinear wave equation with a supercritical nonlinearity. In particular, if we indicate by P_n the orthoprojector constructed with the first n eigenfunctions of A and we set

$$U_0^n = (u_0^n, u_1^n) = (P_n u_0, P_n u_1) \in \mathcal{V}_{0n} := (P_n H)^2, \tag{3.1}$$

then it is not difficult to prove that the corresponding approximating solution $U^n(t) = (u^n(t), u_t^n(t))$ to

$$\varepsilon u_{tt}^n + u_t^n + A(Au^n + P_n f(u^n)) = g^n := P_n g, \tag{3.2}$$
$$u^n|_{t=0} = u_0^n, \quad u_t^n|_{t=0} = u_1^n, \tag{3.3}$$

also satisfies the analogue of (2.17), namely,

$$\|U^n(t)\|^2_{\mathcal{X}_0^\varepsilon} + \int_t^\infty \|u_t^n(s)\|^2_{V'}\, \mathrm{d}s \le C \|U^n(\tau)\|^2_{\mathcal{X}_0^\varepsilon} e^{-\kappa(t-\tau)} + C(1 + \|g\|^2_H), \quad \forall t \ge \tau \ge 0. \tag{3.4}$$

From now on, we restrict ourselves to consider only those of energy bounded solutions which can be obtained as a weak limit of the corresponding Galerkin approximations.

Definition 3.1. *An energy bounded solution $U(t) := (u(t), u_t(t))$ of problem* (1.1) *is an energy solution if it can be obtained as a weak limit of a subsequence of solutions $U_n(t)$ to the Galerkin approximation equations* (3.2) *which satisfy* (3.3).

Since the uniqueness of the energy solutions is not known so far, we use the so-called trajectory approach developed in [11] in order to describe the long-time behavior of such solutions. To this end, we first need to define the trajectory phase space associated with energy solutions of problem (1.1) and the trajectory dynamical system on it.

Definition 3.2. *We define the trajectory phase space $K_\varepsilon^+ \in L^\infty(\mathbb{R}_+, \mathcal{X}_0^\varepsilon)$ as a set of all energy solutions U of problem* (1.1) *associated with all possible initial data $U(0) \in \mathcal{X}_0^\varepsilon$. Namely, we set*

$$K_\varepsilon^+ := \Big\{ U \in L^\infty(\mathbb{R}_+; \mathcal{X}_0^\varepsilon) \ : \ \exists \{U^{n_k}(t)\} \text{ solving } (3.2)-(3.3) \text{ such that} \\ U(0) = [\mathcal{X}_0^\varepsilon]^w - \lim_{k\to\infty} U^{n_k}(0) \text{ and } U = \Theta^+ - \lim_{k\to\infty} U^{n_k} \Big\} \tag{3.5}$$

and we endow K_ε^+ with the topology of $\Theta^+ := [L^\infty_{loc}([0,\infty), \mathcal{X}_0^\varepsilon)]^{w^*}$ (the weak star topology of $L^\infty_{loc}([0,\infty), \mathcal{X}_0^\varepsilon)$). Then, the time translation semigroup

$$\mathbb{T}_\ell : K_\varepsilon^+ \to K_\varepsilon^+, \qquad (\mathbb{T}_\ell u)(t) = u(t+\ell), \tag{3.6}$$

is well defined for $\ell \geq 0$. The semigroup $\mathbb{T}_\ell$ acting on K_ε^+ endowed by the above defined topology is called the trajectory dynamical system associated with equation (1.1).

We recall that a sequence $\{V^n\} \subset L^\infty(\mathbb{R}_+; \mathcal{X}_0^\varepsilon)$ converges to V in Θ^+ if, for every $T \geq 0$, $V^n \to V$ weakly star in $L^\infty((T, T+1); \mathcal{X}_0^\varepsilon)$ as n goes to ∞. Similarly, we can endow $L^\infty(\mathbb{R}; \mathcal{X}_0^\varepsilon)$ with the weak star local topology $\Theta := [L^\infty_{loc}(\mathbb{R}, \mathcal{X}_0^\varepsilon)]^{w^*}$. The obtained topological spaces are Hausdorff and Fréchet-Urysohn with a countable base of open sets (see [11, Chap. XII]).

In order to be able to speak about the attractor of the trajectory dynamical system $(\mathbb{T}_\ell, K_\varepsilon^+)$ (i.e., the trajectory attractor of equation (1.1)), we also need to define the class of *bounded* sets in a proper way. We note that, in contrast to [11], only the energy bounded solutions which can be obtained through the Galerkin limit are included in K_ε^+ (and that difference is crucial for what follows). By this reason, we need, in addition, to introduce (following [37]) the so-called M-functional on the space K_ε^+:

$$M_u^\varepsilon(t) := \inf \Big\{ \liminf_{k\to\infty} \|U^{n_k}(t)\|_{\mathcal{X}_0^\varepsilon} \ : \ U = \Theta^+ - \lim_{k\to\infty} U^{n_k}, \ U(0) = [\mathcal{X}_0^\varepsilon]^w - \lim_{k\to\infty} U^{n_k}(0) \Big\}, \tag{3.7}$$

where the infimum is taken over all the sequences $\{U^{n_k}(t)\}_{k\in\mathbb{N}}$ of Faedo-Galerkin approximations which Θ^+-converge to the given solution U.

Recalling now [37, Cor. 1.1], we can easily prove the following properties of the $M-$energy functional, i.e., for any $U \in K_\varepsilon^+$ we have

$$M_u^\varepsilon(t) < \infty, \quad \|U(t)\|_{\mathcal{X}_0^\varepsilon} \leq M_u^\varepsilon(t), \quad M^\varepsilon_{\mathbb{T}_\ell u}(t) \leq M_u^\varepsilon(t+\ell), \tag{3.8}$$

$$M_u^\varepsilon(t)^2 + \int_t^\infty \|u_t(s)\|^2_{V'} ds \leq C M_u^\varepsilon(\tau)^2 e^{-\kappa(t-\tau)} + C_0(1 + \|g\|_H^2), \quad \forall t \geq \tau \geq 0. \tag{3.9}$$

We can now say that a set $B \subset K_\varepsilon^+$ is $M-$bounded if

$$\sup_{U\in B} M_u^\varepsilon(0) < \infty \tag{3.10}$$

and recall the definition of the trajectory attractor associated with (1.1).

Definition 3.3. A set $\mathcal{A}_\varepsilon^{tr} \subset K_\varepsilon^+$ is a trajectory attractor associated with energy solutions of equation (1.1) (i.e., the global attractor of the trajectory dynamical system $(\mathbb{T}_\ell, K_\varepsilon^+)$) if:

I) the set $\mathcal{A}_\varepsilon^{tr}$ is compact in K_ε^+ (endowed by the Θ^+ topology);

II) it is strictly invariant: $\mathbb{T}_\ell \mathcal{A}_\varepsilon^{tr} = \mathcal{A}_\varepsilon^{tr}$, $\ell \geq 0$;

III) for every M-bounded set $B \subset K_\varepsilon^+$ and every neighborhood $\mathcal{O}(\mathcal{A}_\varepsilon^{tr})$ of $\mathcal{A}_\varepsilon^{tr}$ (again in the topology of Θ^+), there exists $T = T(B, \mathcal{O})$ such that (attraction property)

$$\mathbb{T}_\ell B \subset \mathcal{O}(\mathcal{A}_\varepsilon^{tr}), \quad \forall \ell \geq T.$$

We can now state the existence of the trajectory attractor which can be proven arguing as in [37, Thm. 1.1].

Theorem 3.4. *Let (2.6)-(2.8) and (2.10) hold. Then the semigroup $\mathbb{T}_\ell$ acting on K_ε^+ possesses the trajectory attractor $\mathcal{A}_\varepsilon^{tr}$ characterized as follows*

$$\mathcal{A}_\varepsilon^{tr} = \Pi_+ \mathcal{K}_\varepsilon, \tag{3.11}$$

where $\mathcal{K}_\varepsilon \in L^\infty(\mathbb{R}; \mathcal{X}_0^\varepsilon)$ is the set of all the complete $\mathcal{X}_0^\varepsilon$-bounded solutions to P_ε which can be obtained as a Faedo-Galerkin limit, while Π_+ is the projection onto $L^\infty(\mathbb{R}_+; \mathcal{X}_0^\varepsilon)$. More precisely, $U \in \mathcal{K}_\varepsilon$ if and only if there exist $\{t_k\}$ such that $t_k \searrow -\infty$ and $\{U^{n_k}(t)\}$ such that, for $t \geq t_k$,

$$\varepsilon u_{tt}^{n_k} + u_t^{n_k} + A(Au^{n_k} + P_{n_k} f(u^{n_k})) = g^{n_k}, \tag{3.12}$$

$$u^{n_k}|_{t=t_k} = u_0^k, \quad u_t^{n_k}|_{t=t_k} = u_1^k, \tag{3.13}$$

with

$$U^{n_k}(t_k) \in (P_{n_k} H)^2, \qquad \|U^{n_k}(t_k)\|_{\mathcal{X}_0^\varepsilon} \leq C, \qquad U = \Theta - \lim_{k \nearrow \infty} U^{n_k}, \tag{3.14}$$

where $C > 0$ is independent of k.

The proof of this theorem repeats word by word the proof of [37, Thm. 1.1] and, for this reason, is omitted. Still arguing as in [37], we can deduce the following corollaries. In the statements, if X, Y are Banach spaces, $C_b(X; Y)$ will denote the Banach space of all continuous *and bounded* functions from X to Y, endowed with the supremum norm.

Corollary 3.5. *Let $B \subset K^+$ an M-bounded set. Then, for every $T \in \mathbb{R}_+$ and every $\beta \in (0,1]$, the following convergence holds*

$$\lim_{\ell \to \infty} \operatorname{dist}_{\mathcal{L}_\beta(\ell, T+\ell)}(B|_{[\ell, T+\ell]}, \mathcal{A}^{tr}|_{[\ell, T+\ell]}) = 0, \tag{3.15}$$

where

$$\mathcal{L}_\beta(\ell, T+\ell) = C([\ell, T+\ell], [D(A^{(1-\beta)/2}) \cap L^{p+4-\beta}(\Omega)] \times \sqrt{\varepsilon} D(A^{-(1+\beta)/2})). \tag{3.16}$$

Here we recall the definition of the Hausdorff semidistance, namely

$$\operatorname{dist}_{\mathcal{L}}(\mathcal{B}_1, \mathcal{B}_2) := \sup_{u \in \mathcal{B}_1} \inf_{v \in \mathcal{B}_2} d_{\mathcal{L}}(u, v), \tag{3.17}$$

where $\mathcal{L}$ is some given metric space with distance $d_{\mathcal{L}}$ and $\mathcal{B}_j \subset \mathcal{L}$, $j = 1, 2$.

Corollary 3.6. *Let $U \in \mathcal{K}_\varepsilon$. Then, we have*

$$\int_{-\infty}^{\infty} \|u_t(s)\|_{V'}^2 ds \leq c(1 + \|g\|_H^2), \quad u_{tt} \in C_b(\mathbb{R}, D(A^{-2})). \tag{3.18}$$

Thus, for every $\beta > 0$, there hold

$$u_t \in C_b(\mathbb{R}, D(A^{-(1+\beta)/2})), \quad \lim_{t \to \pm\infty} \|A^{-(1+\beta)/2} u_t(t)\|_H^2 = 0, \tag{3.19}$$

and we also have the convergence to the set of equilibria $\mathcal{R}$

$$\operatorname{dist}_{\left(D(A^{(1-\beta)/2}) \cap L^{p+4-\beta}(\Omega)\right) \times \sqrt{\varepsilon} D(A^{-(1+\beta)/2})}(U(t), \mathcal{R}) \to 0, \tag{3.20}$$

as t goes to ∞, for any $\beta \in (0,1]$.

Remark 3.7. The choice of defining a trajectory dynamical system by selecting those solutions which are limits of Galerkin approximations excludes other possible solutions $(u, u_t) \in L^\infty(\mathbb{R}_+, \mathcal{X}_0^\varepsilon)$ which satisfy the equation (1.1) in the sense of distributions but cannot be obtained in that way. Actually, nothing is known about such "pathological" solutions and theoretically they may exist and even may not be dissipative (that is, they may not satisfy the energy inequality). We remind that we cannot exclude that this might happen even in dimension two when f is allowed to have a supercubical growth. Thus, in order to be able to deal with attractors (no matter using the trajectory or multi-valued semigroup approaches), one should restrict the admissible set of solutions. To do that, there

are at least two alternative ways. The first one is to consider only the solutions which satisfy some weakened form of energy inequality, like the 3D Navier-Stokes equations (see, e.g., [11] and references therein). The second one (used in [37] and in this paper) is to consider only the solutions which can be obtained by the Galerkin approximations (see [30, Rem. 6.2] for more details). We only mention here that both of them present some drawbacks. In particular, the class of energy solutions may depend on some artificial constants in the energy inequality in the former approach, while it may depend on the choice of a Galerkin basis in the latter one. Nonetheless, the second approach has an advantage which is crucial in the present case, namely, it allows to justify the further energy-like inequalities for the energy solutions. In fact, it is completely unclear how to verify most of the results of this paper using the first method. One more drawback is that both the mentioned approaches destroy the concatenation property of energy solutions and, up to the moment, no reasonable way to preserve this property and exclude the "pathological" non-dissipative solutions is known. Thus, the concatenation property seems to be an extremely restrictive assumption which can be verified only in relatively simple cases (usually, when the non-uniqueness is simply due to the presence of non-Lipschitz nonlinearities). For this reason, in the multi-valued approach, one usually needs (following [3], see also [32] and its references) to replace the semigroup *identity* by the semigroup *inclusion* and the constructed attractor will be also only *semi*-invariant with respect to the semigroup itself. This is exactly the approach applied in [33] to analyze equation (1.1). However, even though the trajectory approach is formally equivalent, the trajectory attractor remains strictly invariant even without the concatenation property (see Definition 3.3) and that makes it more convenient for concrete applications.

4 Backward smoothness of complete trajectories

In this section, we prove that any $U = (u, u_t) \in \mathcal{K}_\varepsilon$ (see Theorem 3.4) is backward smooth, i.e., $U(t) \in \mathcal{V}_3^\varepsilon$ if t is small enough. This kind of result is similar to [37, Thm. 2.1]; however, here we follow an alternative strategy which makes use of stationary solutions. Actually, in [37], the globally defined trajectory was compared with the corresponding trajectory of the limit evolution equation obtained by setting $\varepsilon = 0$. Here, instead, we first find a solution v to an auxiliary equation and we prove that v is backward smooth. Then, we show that actually $v(t) \equiv u(t)$ for all $t \le T$, when T is small enough. A crucial point of the argument is the construction of a smooth solution which will then be viewed as the nonvanishing part of v. This is the content of the following

Theorem 4.1. *Let $U = (u, u_t) \in \mathcal{K}_\varepsilon$. Then, for every $\sigma > 0$, there exist $T = T(\sigma, u) < 0$ and a function $\tilde{u} = \tilde{u}_\sigma \in C^\infty(\mathbb{R}_-; H^4)$ such that:*

1) for every $\beta > 0$, there exists $c > 0$ independent of σ such that, for every $t \le T$,

$$\|u(t) - \widetilde{u}(t)\|_{H^{1-\beta}} + \|u_t(t) - \widetilde{u}_t(t)\|_{H^{-\beta}} \le c\sigma; \tag{4.1}$$

2) there exists $C > 0$, which is independent of σ and t, such that

$$\|\tilde{u}\|_{C^2(\mathbb{R}_-;H^4)} \le C(1 + \|g\|_H); \tag{4.2}$$

3) for each $m \in \mathbb{N}$, $m \ge 1$, there exist $C_m > 0$ and $\kappa > 0$, which are independent of σ and t, such that

$$\|\partial_t^m \tilde{u}(t)\|_{H^4} \le C_m \sigma^\kappa, \qquad \forall\, t \le T. \tag{4.3}$$

4) $\tilde{u}$ solves

$$\varepsilon \tilde{u}_{tt} + \tilde{u}_t + A^2 \tilde{u} + Af(\tilde{u}) = g + \phi(t), \tag{4.4}$$

where ϕ is a suitable function such that

$$\|\phi(t)\|_H + \|\phi_t(t)\|_H \le C\sigma^\kappa, \qquad \forall\, t \le T. \tag{4.5}$$

Proof. Using (3.19) and (3.20), we see that, for every $\sigma > 0$ and every $S \in \mathbb{N}$, there exists $T = T(\sigma, u, S) < 0$ such that, for every $s \le T$, there is an equilibrium u_s which satisfies, for some $\beta \in (0, 1]$,

$$\sup_{t \in [s, s+S]} \|u(t) - u_s\|_{H^{1-\beta}} \le \sigma. \tag{4.6}$$

In fact, fixing say $S = 2$ is sufficient for the proof.

Let us check now that

$$\|u_{s+S/2} - u_s\|_{H^4} \le C\sigma^\kappa \tag{4.7}$$

for some positive C and κ independent of σ and s. Indeed, since u_s and $u_{s+S/2}$ are equilibria and $g \in H$, from the elliptic regularity we have that $u_s, u_{s+S/2} \in H^4$. On the other hand, we have

$$A^2(u_s - u_{s+S/2}) = -A(f(u_s) - f(u_{s+S/2})).$$

Thus, recalling that $f \in C^3(\mathbb{R};\mathbb{R})$, we recover

$$\|u_s - u_{s+S/2}\|_{H^5} \le C. \tag{4.8}$$

On the other hand, by definition of u_s and $u_{s+S/2}$, we have, for $\beta \in (0,1]$,

$$\|u(s + S/2) - u_s\|_{H^{1-\beta}} + \|u(s + S/2) - u_{s+S/2}\|_{H^{1-\beta}} \le 2\sigma. \tag{4.9}$$

This yields

$$\|u_s - u_{s+S/2}\|_{H^{1-\beta}} \le 2\sigma, \tag{4.10}$$

which, together with (4.8) and interpolation, entails (4.7).

We are now ready to construct the desired function $\tilde{u}(t)$. Fix $S = 2$ and introduce a cut-off function $\theta \in C_0^\infty(\mathbb{R})$ such that $\theta(t) \equiv 0$ for $t \le 0$, $\theta(t) \equiv 1$, for $t \ge 1$, $0 \le \theta(t) \le 1$. Then, for any $\sigma > 0$ and any $N \in \mathbb{N}$, define a function $\tilde{u}(t)$ on the interval $t \in [T-N, T-N+1]$, $T = T(\sigma)$, by the following formula

$$\tilde{u}(t) := \theta(t - T + N)u_{T-N+1} + (1 - \theta(t - T + N))u_{T-N}. \tag{4.11}$$

This function is clearly smooth with respect to t and fulfills (4.2) and (4.3) (cf. (4.7)). Moreover, since both equilibria u_{T-N+1} and u_{T-N} are close to $u(t)$ on the interval $t \in [T-N, T-N+1]$ (see (4.6)), we also have, for all $t \le T$,

$$\|u(t) - \tilde{u}(t)\|_{H^{1-\beta}} \le 2\sigma. \tag{4.12}$$

Thus, on account of (3.19) and (4.10), we infer that (4.1) holds.

To conclude the proof, it remains to show (4.4) and (4.5). By (4.3), this is equivalent to check that the function

$$\tilde{\phi} := A^2\tilde{u}(t) + Af(\tilde{u}) - g \tag{4.13}$$

is uniformly small. To this end, observe that

$$\begin{aligned}\tilde{\phi}(t) &= A^2 u_{T-N} + \theta(t - T + N)A^2[u_{T-N+1} - u_{T-N}] \\ &\quad + Af(u_{T-N} + \theta(t - T + N)[u_{T-N+1} - u_{T-N}]) - g \\ &= \theta(t - T + N)A^2[u_{T-N+1} - u_{T-N}] \\ &\quad + A[f(u_{T-N} + \theta(t - T + N)[u_{T-N+1} - u_{T-N}]) - f(u_{T-N})]\end{aligned} \tag{4.14}$$

and, using (4.7) and the C^3-regularity of f, it is not difficult to conclude that

$$\|\tilde{\phi}(t)\|_H + \|\tilde{\phi}_t(t)\|_H \le C\sigma^\kappa. \tag{4.15}$$

This inequality and (4.3) imply (4.4) and (4.5). ∎

Now, for $L > 0$ large enough (to be chosen below), we look for a solution v to the equation

$$\varepsilon v_{tt} + v_t + A^2 v + Af(v) + LA^{-1}v = h(t), \tag{4.16}$$

where

$$h(t) = g + LA^{-1}u(t). \tag{4.17}$$

We recall that $U = (u, u_t) \in \mathcal{K}_\varepsilon$ is given.

Observe that (cf. Corollary 3.6)

$$\|h(T)\|_H^2 + \int_T^{T+1} \|A^{(1-\beta)/2} h_t(t)\|_H^2 dt \le C_L(1 + \|g\|_H^2),$$
$$h_t \in C_b(\mathbb{R}; D(A^{(1-\beta)/2})), \quad \lim_{t\to-\infty} \|h_t(t)\|_{D(A^{(1-\beta)/2})} = 0,$$

for any $\beta \in (0,1]$.

Let us prove the following (compare with [37, Lemma 2.1])

Theorem 4.2. *Let $U = (u, u_t) \in \mathcal{K}_\varepsilon$ be given. Then, for a sufficiently large $L > 0$, there exists a time $T = T(u, \varepsilon, L) < 0$ such that equation (4.16) possesses a unique strong backward solution $V \in L^\infty(-\infty, T; \mathcal{V}_3^\varepsilon)$ satisfying*

$$\sqrt{\varepsilon}\|v_{tt}(t)\|_H + \|v_t(t)\|_{H^2} + \|v(t)\|_{H^4} \le Q_L(\|g\|_H), \qquad \forall\, t \le T, \tag{4.18}$$

where the positive monotone function $Q_L(\cdot)$ is independent of ε. Moreover, we have

$$\lim_{t\to-\infty} \|v_t(t)\|_{L^\infty(\Omega)} = 0. \tag{4.19}$$

Proof. We proceed as in [37, proof of Lemma 2.1] with some modifications introduced by Theorem 4.1. We let $\sigma > 0$ (possibly small enough) and look for a v of the form

$$v(t) = \tilde{u}(t) + w(t), \tag{4.20}$$

where we recall that $\tilde{u}$ depends on σ.

Consequently, w must solve the equation

$$\varepsilon w_{tt} + w_t + A^2 w + A(f(\tilde{u} + w) - f(\tilde{u})) + LA^{-1}w = \tilde{h}(t), \tag{4.21}$$

where

$$\tilde{h}(t) = LA^{-1}(u - \tilde{u})(t) - \phi(t). \tag{4.22}$$

We will solve equation (4.21) by means of the inverse function Theorem (cf., e.g., [1, Thm. 2.1.2]).

Thus, we take $T < 0$ small enough and consider the Banach space $\Psi_b := H^1_{loc}((-\infty, T]; H)$ which is endowed with the uniformly local norm

$$\|G\|_{H^1_b((-\infty,T];H)} := \sup_{t\in(-\infty,T-1]} \|G\|_{H^1((t,t+1);H)}. \tag{4.23}$$

Then, it is easy to check that Ψ_b is continuously embedded into $C_b((-\infty, T]; H)$. We also define

$$\Phi_b := \big\{ w \in C_b^2((-\infty,T]; H) \cap C_b^1((-\infty,T]; H^2) \cap C_b((-\infty,T]; H^4) :\ \varepsilon w_{tt} + A^2 w \in \Psi_b \big\}. \tag{4.24}$$

Also the space Φ_b will be endowed with the natural (graph) norm. Thus, we can define the operator $\mathcal{T} : \Phi_b \to \Psi_b$ given by

$$\mathcal{T}_L : w \mapsto \varepsilon w_{tt} + w_t + A^2 w + A(f(\tilde{u} + w) - f(\tilde{u})) + LA^{-1}w.$$

Then, we observe that, recalling (4.1) and (4.5), there exists $T = T(\sigma, u) \in \mathbb{R}$ such that

$$\|\tilde{h}\|_{H^1((t,t+1);H)} \le C_L(\sigma + \sigma^\kappa), \qquad \forall\, t \le T. \tag{4.25}$$

Thus we work in a neighborhood of 0 and we consider the linear operator $\mathcal{T}_L'(0)$ from Φ_b to Ψ_b given by

$$\mathcal{T}_L'(0)W = \varepsilon W_{tt} + W_t + A^2 W + A(f'(\tilde{u})W) + LA^{-1}W.$$

It then suffices to show that $\mathcal{T}_L'(0)$ is invertible. Thus we consider the variation equation at $w = 0$, namely,

$$\varepsilon W_{tt} + W_t + A^2 W + A(f'(\tilde{u})W) + LA^{-1}W = G(t). \tag{4.26}$$

Here, we will assume that G is a given function of Ψ_b. We will now prove that (4.26) has a *unique* solution $W \in \Phi_b$, that is, $\mathcal{T}_L'(0)W = G$, for T small enough and L large enough.

Let us proceed formally by multiplying equation (4.26) by $A^{-1}(W_t + \alpha W)$, $\alpha > 0$. We get

$$\begin{aligned} &\frac{d}{dt}E_W + 2(1-\alpha\varepsilon)\|A^{-1/2}W_t\|_H^2 + 2\alpha\|A^{1/2}W\|_H^2 + 2\alpha L\|A^{-1}W\|_H^2 + 2\alpha(f'(\tilde{u})W, W) \\ &= 2(A^{-1/2}G, A^{-1/2}(W_t + \alpha W)) + (f''(\tilde{u})\tilde{u}_t, W^2), \end{aligned} \tag{4.27}$$

where

$$\begin{aligned} E_W &= \varepsilon\|A^{-1/2}W_t\|_H^2 + \|A^{1/2}W_H\|^2 + L\|A^{-1}W\|_H^2 \\ &\quad + (f'(\tilde{u})W, W) + 2\alpha\varepsilon(A^{-1/2}W, A^{-1/2}W_t) + \alpha\|A^{-1/2}W\|_H^2. \end{aligned} \tag{4.28}$$

Observe that, recalling (2.7) and using interpolation and Young's inequality, we have

$$\begin{aligned} &-(f'(\tilde{u})W, W) \le \lambda\|W\|_H^2 \le c_1\lambda\|W\|_{H^1}^{4/3}\|W\|_{H^{-2}}^{2/3} \\ &\le c_2\lambda\|A^{1/2}W\|_H^{4/3}\|A^{-1}W\|_H^{2/3} \le \frac{1}{2}\left(\|A^{1/2}W\|_H^2 + c_3\lambda^3\|A^{-1}W\|_H^2\right), \end{aligned} \tag{4.29}$$

where c_j, $j = 1, 2, 3$, are positive constants independent of W.

Then, recalling (2.7), we can choose $L > c_3\lambda^3$ so that

$$(f'(\tilde{u})W, W) + \frac{1}{2}\left(\|A^{1/2}W\|_H^2 + L\|A^{-1}W\|_H^2\right) \ge 0. \tag{4.30}$$

Picking α small enough (but independent of ε and L), we then deduce

$$C_1^{-1}\left(\varepsilon\|A^{-1/2}W_t\|_H^2 + \|A^{1/2}W\|_H^2\right) \le E_W \le C_1\left(\varepsilon\|A^{-1/2}W_t\|_H^2 + \|A^{1/2}W\|_H^2\right), \tag{4.31}$$

for some $C_1 > 1$.

Consequently, from (4.27) we infer the inequality

$$\frac{d}{dt}E_W + kE_W \le C\|A^{-1/2}G\|_H^2 + (f''(\tilde{u})\tilde{u}_t, W^2) - \frac{\alpha}{2}\|A^{1/2}W\|_H^2, \tag{4.32}$$

for some $k > 0$. On the other hand, for σ (and T) small enough, we have, due to (4.3),

$$(f''(\tilde{u})\tilde{u}_t, W^2) \le \frac{\alpha}{2}\|A^{1/2}W\|_H^2. \tag{4.33}$$

Thus we have

$$\frac{d}{dt}E_W + kE_W \le C\|A^{-1/2}G\|_H^2.$$

Consider now a sequence $\{t_n\} \subset (-\infty, t)$ such that $t_n \to -\infty$. Then Gronwall's inequality gives

$$\begin{aligned} \varepsilon\|A^{-1/2}W_t(t)\|_H^2 + \|A^{1/2}W(t)\|_H^2 &\le \left(\varepsilon\|A^{-1/2}W_t(t_n)\|_H^2 + \|A^{1/2}W(t_n)\|_H^2\right)e^{k(t_n - t)} \\ &\quad + C_L\int_{-\infty}^t e^{-k(t-s)}\|A^{-1/2}G(s)\|_H^2 ds. \end{aligned}$$

Since we are looking for solutions in Φ_b we can let t_n go to $-\infty$ and recover

$$\varepsilon\|A^{-1/2}W_t(t)\|_H^2 + \|A^{1/2}W(t)\|_H^2 \le C_L\int_{-\infty}^t e^{-k(t-s)}\|A^{-1/2}G(s)\|_H^2 ds, \tag{4.34}$$

for all $t \le T$. Hence, in particular, W is necessarily unique.

Let us now set

$$\widetilde{W} = W_t \tag{4.35}$$

and observe that time differentiation of (4.26) gives

$$\varepsilon\widetilde{W}_{tt}+\widetilde{W}_t+A^2\widetilde{W}+A(f'(\tilde{u})\widetilde{W})+LA^{-1}\widetilde{W}=G_t-A(f''(\tilde{u})\tilde{u}_tW). \tag{4.36}$$

Multiplying this equation by $A^{-1}(\widetilde{W}_t+\alpha\widetilde{W})$ and arguing as above, we find

$$\begin{aligned}&\varepsilon\|A^{-1/2}\widetilde{W}_t(t)\|_H^2+\|A^{1/2}\widetilde{W}(t)\|_H^2\\&\leq C_L\int_{-\infty}^t e^{-k(t-s)}\left(\|A^{-1/2}G_t(s)\|_H^2+\|A^{1/2}(f''(\tilde{u}(s))\tilde{u}_t(s)W(s))\|_H^2\right)ds,\end{aligned} \tag{4.37}$$

for all $t\leq T$. On account of (4.3) and (4.34), observe now that

$$\begin{aligned}\|A^{1/2}(f''(\tilde{u}(s))\tilde{u}_t(s)W(s))\|_H&\leq C\|A^{1/2}W(s)\|_H\\&\leq C_L\sup_{t\in(-\infty,T-1]}\|G\|_{L^2((t,t+1);D(A^{-1/2}))}.\end{aligned} \tag{4.38}$$

Therefore, (4.37) and (4.38) yield

$$\varepsilon\|A^{-1/2}W_{tt}(t)\|_H^2+\|A^{1/2}W_t(t)\|_H^2\leq C_L\sup_{t\in(-\infty,T-1]}\|G\|^2_{H^1((t,t+1);D(A^{-1/2}))}, \tag{4.39}$$

and by comparison in (4.26), we also deduce

$$\|A^{3/2}W(t)\|_H\leq C_L\sup_{t\in(-\infty,T-1]}\|G\|_{H^1((t,t+1);D(A^{-1/2}))}. \tag{4.40}$$

We can now multiply equation (4.36) by $\widetilde{W}_t+\alpha\widetilde{W}$ and use the identity

$$\begin{aligned}(A(f'(\tilde{u})\widetilde{W}),\widetilde{W}_t)&=\int_\Omega(\nabla(f'(\tilde{u})\widetilde{W}),\nabla\widetilde{W}_t)\\&=\frac{d}{dt}\int_\Omega f'(\tilde{u})\frac{|\nabla\widetilde{W}|^2}{2}-\int_\Omega f''(\tilde{u})\tilde{u}_t\frac{|\nabla\widetilde{W}|^2}{2}\\&\quad-\int_\Omega\nabla\cdot[f''(\tilde{u})\widetilde{W}\nabla\tilde{u}]\widetilde{W}_t.\end{aligned}$$

This gives the identity

$$\begin{aligned}&\frac{d}{dt}\left[\varepsilon\|\widetilde{W}_t\|_H^2+\|A\widetilde{W}\|_H^2+L\|A^{-1/2}\widetilde{W}\|_H^2+2\alpha\varepsilon(\widetilde{W},\widetilde{W}_t)+\alpha\|\widetilde{W}\|_H^2+\int_\Omega f'(\tilde{u})|\nabla\widetilde{W}|^2\right]\\&+2(1-\alpha\varepsilon)\|\widetilde{W}_t\|_H^2+2\alpha\|A\widetilde{W}\|_H^2+2\alpha L\|A^{-1/2}\widetilde{W}\|_H^2+2\alpha(A^{1/2}(f'(\tilde{u})\widetilde{W}),A^{1/2}\widetilde{W})\\&=2(G_t,\widetilde{W}_t+\alpha\widetilde{W})+\int_\Omega f''(\tilde{u})\tilde{u}_t|\nabla\widetilde{W}|^2+2\int_\Omega\nabla\cdot[f''(\tilde{u})\widetilde{W}\nabla\tilde{u}]\widetilde{W}_t-2(A(f''(\tilde{u})\tilde{u}_tW),\widetilde{W}_t+\alpha\widetilde{W}).\end{aligned} \tag{4.41}$$

Observe that, by (2.7) and (4.2)-(4.3),

$$\int_\Omega f'(\tilde{u})|\nabla\widetilde{W}|^2\leq C\|A^{1/2}\widetilde{W}\|_H^2, \tag{4.42}$$

$$(A^{1/2}(f'(\tilde{u})\widetilde{W}),A^{1/2}\widetilde{W})\leq C\|A^{1/2}\widetilde{W}\|_H^2, \tag{4.43}$$

$$\int_\Omega f''(\tilde{u})\tilde{u}_t|\nabla\widetilde{W}|^2\leq C\|A^{1/2}\widetilde{W}\|_H^2, \tag{4.44}$$

$$\int_\Omega\nabla\cdot[f''(\tilde{u})\widetilde{W}\nabla\tilde{u}]\widetilde{W}_t\leq C\|A^{1/2}\widetilde{W}\|_H\|\widetilde{W}_t\|_H, \tag{4.45}$$

$$\|A(f''(\tilde{u})\tilde{u}_tW)\|_H\leq C\|AW\|_H. \tag{4.46}$$

Next, due to

$$\|A^{1/2}\widetilde{W}\|^2\leq C\|\widetilde{W}\|_{H^2}^{4/3}\|\widetilde{W}\|_{H^{-1}}^{2/3}\leq C\|A\widetilde{W}\|_H^{4/3}\|A^{-1/2}\widetilde{W}\|_H^{2/3},$$

also in this case we can choose L large enough so that

$$\int_\Omega f'(\tilde{u})|\nabla\widetilde{W}|^2 + \frac{1}{2}\left(\|A\widetilde{W}\|_H^2 + L\|A^{-1/2}\widetilde{W}\|_H^2\right) \geq 0.$$

We now indicate by $\widetilde{E}_{\widetilde{W}}$ the expression within square brackets in identity (4.41). Then we can choose α small enough (but independent of ε and L) in such a way that

$$C_2^{-1}\left(\varepsilon\|\widetilde{W}_t\|_H^2 + \|A\widetilde{W}\|_H^2\right) \leq \widetilde{E}_{\widetilde{W}} \leq C_2\left(\varepsilon\|\widetilde{W}_t\|_H^2 + \|A\widetilde{W}\|_H^2\right), \tag{4.47}$$

for some $C_2 > 1$. Then, from (4.41) and (4.43)-(4.46) we deduce the inequality

$$\frac{d}{dt}\widetilde{E}_{\widetilde{W}} + \alpha\widetilde{E}_{\widetilde{W}} \leq C(\|G_t\|_H^2 + \|AW\|_H^2 + \|A^{1/2}W_t\|_H). \tag{4.48}$$

Therefore, owing to (4.47) and recalling (4.39), a further application of Gronwall's inequality to (4.48) yields

$$\varepsilon\|\widetilde{W}_t(t)\|_H^2 + \|A\widetilde{W}(t)\|_H^2 \leq C_L\int_{-\infty}^t e^{-\alpha(t-s)}(\|G_t(s)\|_H^2 + \|AW(s)\|_H^2)ds, \tag{4.49}$$

for all $t \leq T$. This yields, on account of (4.39) and (4.40),

$$\varepsilon\|W_{tt}(t)\|_H^2 + \|AW_t(t)\|_H^2 \leq C_L\|G\|^2_{H_b^1((-\infty,T);H)}, \tag{4.50}$$

and suitable comparison arguments in (4.26) entail that

$$\|W(t)\|_{H^4} \leq C_L\|G\|_{H_b^1((-\infty,T);H)} \tag{4.51}$$

and, correspondingly,

$$\|W_{tt} + A^2W(t)\|_{H_b^1((-\infty,T);H)} \leq C_L\|G\|_{H_b^1((-\infty,T);H)}. \tag{4.52}$$

A priori estimates (4.50) and (4.51)-(4.52) combined with standard arguments allow us to conclude that equation (4.26) has indeed a (unique) solution in Φ_b.

Thus, owing to the inverse function theorem, we conclude that (4.21) has a unique solution w for T small enough such that

$$\|w_t(t)\|_{H^2} + \|w(t)\|_{H^4} \leq C_L(\|g\|_H). \tag{4.53}$$

where C_L is independent of ε (cf. (4.50)).

Moreover, thanks to (4.25), if we fix a neighborhood $\mathcal{W}$ of 0 in the Banach space (4.24), then we can choose T small enough such that $(w(t), w_t(t)) \in \mathcal{W}$ for all $t \leq T$. Thus, recalling (4.20) and Theorem 4.1, we have, in particular,

$$\lim_{t\to-\infty}\|v_t(t)\|_{H^2} = 0 \tag{4.54}$$

and can conclude that we have found a backward strong solution to (4.16) which satisfies (4.18) and, on account of the arbitrariness of σ (cf. also (4.3)), (4.19). ∎

We are now in a position to prove the main result of this section, namely,

Theorem 4.3. *Let $U = (u, u_t) \in \mathcal{K}_\varepsilon$. Then there exists $T = T_u$ such that*

$$U \in C_b((-\infty, T], \mathcal{V}_3^\varepsilon), \tag{4.55}$$

and, for all $t \leq T_u$,

$$\|u(t)\|_{H^4}^2 + \|u_t(t)\|_{H^2}^2 + \varepsilon\|u_{tt}(t)\|_H^2 \leq Q(\|g\|_H). \tag{4.56}$$

PROOF. The proof follows closely [37] with suitable adaptations. Our goal is to prove that

$$U(t) \equiv V(t), \tag{4.57}$$

for all $t \le T$ and some $T < 0$, where V is given by Theorem 4.2. According to Theorem 3.4, we consider a sequence $\{U^{n_k}(t)\}$ for $t \ge t_k$ satisfying (3.12)-(3.13) and such that (3.14) holds. Also, we define

$$v^{n_k}(t) = P_{n_k} v(t), \qquad \forall\, t \le T, \tag{4.58}$$

where T is given by Theorem 4.2, and we observe that, due to Theorem 4.2, we have

$$\lim_{k\to\infty} \|V^{n_k} - V\|_{C_b((-\infty,T];\mathcal{V}_2^\varepsilon)} = 0, \quad \lim_{k\to\infty} \|v^{n_k} - v\|_{C_b((-\infty,T]:C^1(\bar\Omega))} = 0. \tag{4.59}$$

In addition, we have (cf. (4.18))

$$\lim_{k\to\infty} \|v_t^{n_k} - v_t\|_{C_b((-\infty,T]\times\bar\Omega)} = 0. \tag{4.60}$$

Let us now set

$$Z(t) = u(t) - v(t), \quad Z^{n_k}(t) = u^{n_k}(t) - v^{n_k}(t), \tag{4.61}$$

and observe that

$$\varepsilon Z_{tt}^{n_k} + Z_t^{n_k} + A^2 Z^{n_k} + AP_{n_k}(f(v^{n_k} + Z^{n_k}) - f(v^{n_k})) + LA^{-1}Z^{n_k} = h^{n_k}, \tag{4.62}$$

$$Z^{n_k}|_{t=t_k} = u_0^k - P_{n_k} v(t_k), \quad Z_t^{n_k}|_{t=t_k} = u_1^k - P_{n_k} v_t(t_k), \tag{4.63}$$

where

$$h^{n_k}(t) := AP_{n_k}(f(v(t)) - f(v^{n_k}(t))). \tag{4.64}$$

Observe also that, on account of (4.59), there holds

$$\lim_{k\to\infty} \|A^{-1/2} h^{n_k}\|_{C_b((-\infty,T]\times\bar\Omega)} = 0. \tag{4.65}$$

Moreover, thanks to Theorem 3.4 and (4.18), we also have

$$\|(Z^{n_k}(t_k), Z_t^{n_k}(t_k))\|_{\mathcal{V}_0^\varepsilon} \le C, \qquad \forall\, k \in \mathbb{N}. \tag{4.66}$$

If we multiply equation (4.62) by $A^{-1}(Z_t^{n_k} + \alpha Z^{n_k})$, then we obtain

$$\frac{d}{dt} E_{Z^{n_k}} + \alpha E_{Z^{n_k}} = H_{n_k}, \qquad \text{in } (-\infty, T], \tag{4.67}$$

where

$$\begin{aligned} E_{Z^{n_k}} &= \varepsilon\|A^{-1/2} Z_t^{n_k}\|_H^2 + \|A^{1/2} Z^{n_k}\|_H^2 + L\|A^{-1}Z^{n_k}\|_H^2 + 2\alpha\varepsilon(Z^{n_k}, Z_t^{n_k}) \\ &\quad + \alpha\|A^{-1/2}Z^{n_k}\|_H^2 + 2\left(F(v^{n_k} + Z^{n_k}) - F(v^{n_k}) - f(v^{n_k})Z^{n_k}, 1\right) \end{aligned} \tag{4.68}$$

and

$$\begin{aligned} H_{n_k} &:= -(2 - 3\varepsilon)\|A^{-1/2} Z_t^{n_k}\|_H^2 - \alpha\|A^{1/2}Z^{n_k}\|_H^2 - \alpha L\|A^{-1}Z^{n_k}\|_H^2 \\ &\quad + 2\alpha\left(F(v^{n_k} + Z^{n_k}) - F(v^{n_k}) - f(v^{n_k})Z^{n_k} - (f(v^{n_k} + Z^{n_k}) - f(v^{n_k}))Z^{n_k}, 1\right) \\ &\quad + 2\alpha^2\varepsilon(Z^{n_k}, Z_t^{n_k}) + \alpha^2\|Z^{n_k}\|_H^2 + 2(A^{-1/2}h^{n_k}, A^{-1/2}(Z_t^{n_k} + \alpha Z^{n_k})) \\ &\quad + 2\left(f(v^{n_k} + Z^{n_k}) - f(v^{n_k}) - f'(v^{n_k})Z^{n_k}, v_t^{n_k}\right). \end{aligned} \tag{4.69}$$

We can now take advantage of [37, (2.52)-(2.53)] to estimate the above nonlinear terms. Then, also recalling the choice of L and (4.30), we can find some positive constants α_1, C_1 and C_2 (all independent of v^{n_k}, Z^{n_k}, k, L and ε) such that

$$\begin{aligned} H_{n_k} &\le -\frac{\alpha_1}{2}\left(\|A^{1/2}Z^{n_k}\|_H^2 + \|A^{-1/2}Z_t^{n_k}\|_H^2\right) - \frac{\alpha_1 L}{2}\|A^{-1}Z^{n_k}\|_H^2 - 2\alpha_1(|Z^{n_k}|^{p+4}, 1) \\ &\quad + C_1\|A^{-1/2}h^{n_k}\|_H^2 + C_2\|v_t^{n_k}\|_{L^\infty(\Omega)}\left(|Z^{n_k}|^2(1 + |v^{n_k}|^{p+1} + |Z^{n_k}|^{p+1}), 1\right). \end{aligned} \tag{4.70}$$

Using (4.19) and (4.59)-(4.60), we can also find $T' \leq T$ such that

$$H_{n_k}(t) \leq C_1 \|A^{-1/2} h^{n_k}(t)\|_H^2, \qquad \forall\, t \leq T'. \tag{4.71}$$

Then, applying Gronwall's lemma to (4.67) and using [37, (2.51)] and (4.29), we get

$$\begin{aligned} &\varepsilon \|A^{-1/2} Z_t^{n_k}(t)\|_H^2 + \|A^{1/2} Z^{n_k}(t)\|_H^2 \\ &\leq C_3 \left(1 + \|(Z^{n_k}(t_k), Z_t^{n_k}(t_k))\|_{\mathcal{V}_0^\varepsilon}^2\right) e^{-\alpha(t-t_k)} + 2C_1 \int_{t_k}^t e^{-\alpha(t-s)} \|A^{-1/2} h^{n_k}(s)\|_H^2 ds, \end{aligned} \tag{4.72}$$

for all $t \leq T'$, where C_3 is also independent of k. Then, on account of (4.65) and (4.66), we let k go to ∞ and we recover (cf. (4.61))

$$\varepsilon \|A^{-1/2} Z_t(t)\|_H^2 + \|A^{1/2} Z(t)\|_H^2 \leq 0, \qquad \forall\, t \leq T'. \tag{4.73}$$

Thus (4.57) is proven. Estimate (4.56) follows from (4.18) and (4.57). The proof is complete. ∎

We conclude this section by proving that the solution $U = (u, u_t) \in \mathcal{K}$ is unique until it is regular (cf. [37, Thm. 2.2]).

Theorem 4.4. *Let $U = (u, u_t) \in \mathcal{K}_\varepsilon$ satisfy (4.56) for every $t \leq T_u$. Consider another complete energy solution $\tilde{U} = (\tilde{u}, \tilde{u}_t) \in \mathcal{K}_\varepsilon$ such that, for some $\tilde{T} < T_u$ and for all $t \leq \tilde{T} < T_u$,*

$$U(t) \equiv \tilde{U}(t). \tag{4.74}$$

Then we necessarily have

$$U(t) \equiv \tilde{U}(t), \tag{4.75}$$

for all $t \leq T_u$.

Proof. Arguing as in the proof of Theorem 4.3, we consider a sequence of Galerkin solutions $\{\tilde{U}^{n_k}(t)\}$ which approximate $\tilde{U}$ for $t \geq t_k$. Namely, we assume (3.12), (3.13) and (3.14) to be satisfied. Then, we set

$$U^{n_k} = P_{n_k} U, \quad z = u - \tilde{u}, \quad z^{n_k} = u^{n_k} - \tilde{u}^{n_k},$$

where $u^{n_k} = P_{n_k} u$. Thus, similarly to (4.62)-(4.63), we have

$$\varepsilon z_{tt}^{n_k} + z_t^{n_k} + A^2 z^{n_k} + A P_{n_k}(f(\tilde{u}^{n_k} + z^{n_k}) - f(\tilde{u}^{n_k})) + L A^{-1} z^{n_k} = h^{n_k}(t), \tag{4.76}$$

$$z^{n_k}|_{t=t_k} = u_0^k - P_{n_k} \tilde{u}(t_k), \quad z_t^{n_k}|_{t=t_k} = u_1^k - P_{n_k} \tilde{u}_t(t_k), \tag{4.77}$$

where

$$h^{n_k}(t) := A P_{n_k}(f(\tilde{u}(t)) - f(\tilde{u}^{n_k}(t))) + L A^{-1} z^{n_k}. \tag{4.78}$$

for some positive L which will be chosen below.

Observe that (cf. (4.65)),

$$\lim_{k \to \infty} \|A^{-1/2}(h^{n_k} - L A^{-1} z^{n_k})\|_{C_b((-\infty, T_u]; H)} = 0, \quad \|A^{-1/2} h^{n_k}\|_{C_b((-\infty, T_u]; H)} \leq C_4, \tag{4.79}$$

for some $C_4 > 0$ independent of k.

If we multiply equation (4.76) by $A^{-1}(z_t^{n_k} + \alpha z^{n_k})$, then we obtain (cf. (4.67))

$$\frac{d}{dt} E_{z^{n_k}}(t) + \alpha E_{z^{n_k}}(t) = H_{n_k}(t), \qquad \forall\, t \leq T_u. \tag{4.80}$$

Here $E_{z^{n_k}}$ and H_{n_k} are defined as in (4.68) and (4.69), respectively. Thus, recalling (4.70), we can find some positive constants α_2, C_5 and C_6 (all independent of $\tilde{u}^{n_k}$, z^{n_k}, k, L and ε) such that

$$\begin{aligned} H_{n_k} \leq &-\frac{\alpha_2}{2}\left(\|A^{1/2} z^{n_k}\|_H^2 + \|A^{-1/2} z_t^{n_k}\|_H^2\right) - \frac{\alpha_2 L}{2} \|A^{-1} z^{n_k}\|_H^2 - 2\alpha_2(|z^{n_k}|^{p+4}, 1) \\ &+ C_5 \|A^{-1/2} h^{n_k}\|_H^2 + C_6 \|u_t^{n_k}\|_{L^\infty(\Omega)} \left(|z^{n_k}|^2 (1 + |u^{n_k}|^{p+1} + |z^{n_k}|^{p+1}), 1\right), \end{aligned} \tag{4.81}$$

in $(-\infty, T_u]$. Using now (4.56), we get

$$H_{n_k} \leq -\frac{\alpha_2}{2}\left(\|A^{1/2}z^{n_k}\|_H^2 + \|A^{-1/2}z_t^{n_k}\|_H^2\right) - \frac{\alpha_2 L}{2}\|A^{-1}z^{n_k}\|_H^2 - 2\alpha_2(|z^{n_k}|^{p+4},1) \tag{4.82}$$
$$+ C_5\|A^{-1/2}h^{n_k}\|_H^2 + C_7\left(|z^{n_k}|^2(1+|z^{n_k}|^{p+1}),1\right),$$
$$\leq -\frac{\alpha_2}{2}\left(\|A^{1/2}z^{n_k}\|_H^2 + \|A^{-1/2}z_t^{n_k}\|_H^2\right) - \frac{\alpha_2 L}{2}\|A^{-1}z^{n_k}\|_H^2 - \alpha_2(|z^{n_k}|^{p+4},1)$$
$$+ C_5\|A^{-1/2}h^{n_k}\|_H^2 + C_8\|z^{n_k}\|_H^2.$$

Using (4.29) and Young's inequality, we can then choose L large enough so that

$$H_{n_k}(t) \leq C_5\|A^{-1/2}h^{n_k}(t)\|_H^2, \qquad \forall\, t \leq T_u. \tag{4.83}$$

Applying Gronwall's inequality to (4.80) we obtain, thanks to (4.83),

$$E_{z^{n_k}}(t) \leq E_{z^{n_k}}(t_k)e^{-\alpha(t-t_k)} + C_5\int_{t_k}^{t} e^{-\alpha(t-s)}\|A^{-1/2}h^{n_k}(s)\|_H^2 ds, \qquad \forall\, t \leq T_u. \tag{4.84}$$

We now let k go $+\infty$ in (4.84) and recalling (4.74) and (4.79), we obtain

$$\|Z(t)\|_0 \leq C_5 L^2 \int_{\tilde{T}}^{t} e^{-\alpha(t-s)}\|A^{-3/2}z(s)\|_H^2 ds, \qquad \forall\, t \in [\tilde{T}, T_u]. \tag{4.85}$$

Finally, a further application of Gronwall's lemma to (4.85) entails (4.75). ∎

5 Smoothness of global attractors

We can now show that, under suitable conditions, the trajectory attractor $\mathcal{A}_\varepsilon^{tr}$ (see (3.11)) consists of strong solutions. Let us first analyze the two-dimensional case in the case of cubic controlled growth. Namely, we take $f \in C^3(\mathbb{R};\mathbb{R})$ with $f(0) = 0$ satisfying (2.6), (2.7) and

$$f''' \in L^\infty(\mathbb{R};\mathbb{R}). \tag{5.1}$$

We know that, for any $\varepsilon > 0$, problem P_ε generates a (strongly continuous) semigroup $S_\varepsilon(t)$ on $\mathcal{V}_2^\varepsilon$ (see [26, Thm. 2.2], cf. also [26, Rem. 2.1]), i.e., the trajectories are quasi-strong solutions. Moreover, the dynamical system $(\mathcal{V}_2^\varepsilon, S_\varepsilon(t))$ possesses a global attractor $\mathbb{A}_\varepsilon$ which is bounded in $\mathcal{V}_3^\varepsilon$ (cf. [26, Thm. 4.1]). On the other hand, in the case of energy solutions, we also know that P_ε generates a (strongly continuous) semigroup $\tilde{S}_\varepsilon(t)$ on $\mathcal{V}_0^\varepsilon$ which has the global attractor $\mathcal{A}_\varepsilon$ (see [26, Sec. 6]). Thus, on account of Theorem 3.4, we have that $\mathcal{K}$ coincides with the set of all the complete energy bounded solutions and

$$\mathcal{A}_\varepsilon \equiv \Pi_0\mathcal{A}_\varepsilon^{tr}, \tag{5.2}$$

where $\Pi_0 U(t) = U(0)$.

It is clear that $\mathbb{A}_\varepsilon \subset \mathcal{A}_\varepsilon$, while the validity of the opposite inclusion was an open question so far. Indeed we are now ready to prove the following

Theorem 5.1. *Let* (2.6), (2.7), (2.10) *and* (5.1) *hold. Then*

$$\mathbb{A}_\varepsilon \equiv \mathcal{A}_\varepsilon. \tag{5.3}$$

Proof. Let us consider first the set $\mathbb{K}_\varepsilon$ of all the complete bounded strong solutions so that $\mathbb{A}_\varepsilon \equiv \Pi_0\mathbb{K}_\varepsilon$. It is clear that $\mathbb{K}_\varepsilon \subset \mathcal{K}_\varepsilon$. Let us consider now a bounded complete energy solution U to (2.11). Then, by Theorem 4.3, there exists a time $T = T_u$ such that $U(t) \in \mathcal{V}_3^\varepsilon$ for all $t \leq T$ and satisfies a bound like (4.56). On the other hand, we know that there exists a unique strong solution $\tilde{U}$ to (2.11) bounded on $[T, +\infty)$ and such that $\tilde{U}(T) = U(T)$. Thus we can construct a complete strong solution

$$U^*(t) = \begin{cases} \tilde{U}(t), & t > T, \\ U(t), & t \leq T, \end{cases} \tag{5.4}$$

which is bounded on $\mathbb{R}$ in the $\mathcal{V}_3^\varepsilon$-norm. Thus, Theorem 4.4 implies that $U \equiv U^*$. Therefore, we have that $\mathcal{K}_\varepsilon \subset \mathbb{K}_\varepsilon$ and the proof is complete. ∎

In the three-dimensional case or in two dimensions with supercubical growth, we cannot say more than Theorem 3.4 about the existence of global solutions without making a restriction on ε. More precisely, we have the following result (see [25, Thm. 2.4])

Theorem 5.2. *Let* (2.6)-(2.8) *and* (2.10) *hold. Then, there exist* $\varepsilon_0 > 0$ *and a nonincreasing positive function* $R : (0, \varepsilon_0) \to (0, +\infty)$ *with the property*

$$\lim_{\varepsilon \searrow 0} R(\varepsilon) = +\infty, \tag{5.5}$$

such that, for every $\varepsilon \in (0, \varepsilon_0)$ *and every initial condition* $U_0 = (u_0, u_1) \in \mathcal{V}_1$ *satisfying*

$$\|U_0\|_{\mathcal{V}_1^\varepsilon} \le R(\varepsilon), \tag{5.6}$$

there exists a (unique) global weak solution $U = (u, u_t)$ *to problem* P_ε *such that*

$$\|U(t)\|_{\mathcal{V}_1^\varepsilon} \le Q(\|U_0\|_{\mathcal{V}_1^\varepsilon}) e^{-\kappa t} + Q(\|g\|_H), \quad \forall\, t \ge 0, \tag{5.7}$$

for some $\kappa > 0$ *and some positive increasing monotone function* Q *both independent of* ε*.*

Following [37], we consider the $\mathcal{V}_1^\varepsilon$-ball $B(R(\varepsilon))$ of radius $R(\varepsilon)$ centered at 0 and define the solving operator

$$S_\varepsilon(t) : B(R(\varepsilon)) \to \mathcal{V}_1^\varepsilon, \qquad U(t) = S_\varepsilon(t) U(0), \quad \forall\, t \ge 0.$$

Thanks to estimate (5.7), we have

$$\|S_\varepsilon(t)(B(R(\varepsilon)))\|_{\mathcal{V}_1^\varepsilon} \le \Lambda, \quad \forall\, t \ge 0,$$

for an appropriate positive quantity Λ (clearly also depending on $\|g\|_H$). Then, we set

$$\mathbb{B}_\varepsilon := \left[\bigcup_{t \ge 0} S_\varepsilon(t)(B(R(\varepsilon))) \right]_{\mathcal{V}_1^\varepsilon}, \tag{5.8}$$

where $[\cdot]_{\mathcal{V}_1^\varepsilon}$ denotes the closure in $\mathcal{V}_1^\varepsilon$, and we note that $\mathbb{B}_\varepsilon$ is bounded in $\mathcal{V}_1^\varepsilon$ by Λ. Moreover, $\mathbb{B}_\varepsilon$ is closed and positively invariant by construction. Thus, we can say that $(\mathbb{B}_\varepsilon, S_\varepsilon(t))$ is a dissipative dynamical system.

On the other hand, we know that (see [25, Thm. 2.7]), up to possibly taking a smaller ε_0 and correspondingly modifying the function R, for all $\varepsilon \in (0, \varepsilon_0)$ $(\mathbb{B}_\varepsilon, S_\varepsilon(t))$ possesses a global attractor $\mathbb{A}_\varepsilon$ bounded in $\mathcal{V}_3^\varepsilon$. Consider now the set $\mathbb{K}_\varepsilon$ of all complete and bounded weak solutions taking values in the phase space $\mathbb{B}_\varepsilon$. Thanks to the above considerations, we have that $\mathbb{A}_\varepsilon = \Pi_t \mathbb{K}_\varepsilon$, for any $t \in \mathbb{R}$. Hence, each $U \in \mathbb{K}_\varepsilon$ is indeed a global strong solution. We can now prove

Theorem 5.3. *Let the assumptions of Theorem 5.2 hold. Then there exists* $\varepsilon_1 \in (0, \varepsilon_0]$ *such that, if* $\varepsilon \in (0, \varepsilon_1)$*, then*

$$\mathbb{K}_\varepsilon \equiv \mathcal{K}_\varepsilon, \tag{5.9}$$

where $\mathcal{K}_\varepsilon$ *is defined in Theorem 3.4. In particular, we have*

$$\mathbb{A}_\varepsilon \equiv \Pi_0 \mathcal{A}_\varepsilon^{tr}. \tag{5.10}$$

Proof. It is clear that $\mathbb{K}_\varepsilon \subset \mathcal{K}_\varepsilon$. On the other hand, if we consider a bounded complete energy solution U given by Theorem 3.4 (i.e., an element of the trajectory attractor), then Theorem 4.3 implies the existence of a time $T = T_u$ such that $U(t) \in \mathcal{V}_3^\varepsilon$ for all $t \le T$ and satisfies a bound like (4.56). In particular, this bound entails that, for all $t \le T_u$,

$$\|U(t)\|_{\mathcal{V}_1^\varepsilon} = \left(\|u(t)\|_{H^2}^2 + \varepsilon \|u_t(t)\|_H^2 \right)^{1/2} \le Q_1(\|g\|_H), \tag{5.11}$$

where $Q_1(\cdot)$ is a computable function whose expression is independent of ε. At this point, we restrict ourselves to those $\varepsilon \in (0, \varepsilon_0)$ such that

$$Q_1(\|g\|_H) \le R(\varepsilon). \tag{5.12}$$

By (5.5) and the (decreasing) monotonicity of $R(\cdot)$, this will hold for all ε in some interval $(0, \varepsilon_1)$, where $\varepsilon_1 \leq \varepsilon_0$. By Theorem 5.2, we know that there exists a unique solution $\tilde{U} \in C_b([T, +\infty); \mathcal{V}_1^\varepsilon)$ to (2.11) such that $\tilde{U}(T) = U(T)$. Thus, recalling (5.4), we can construct once more a complete weak solution U^*. Moreover, being $B(R(\varepsilon)) \subset \mathbb{B}_\varepsilon$ by (5.8), this solution is such that

$$U^*(t) \in \mathbb{B}_\varepsilon \quad \forall\, t \in \mathbb{R}.$$

Thus we have that necessarily $U^* \in \mathbb{K}_\varepsilon$. On the other hand, Theorem 4.4 implies that $U \equiv U^*$. Therefore, we have once again $\mathcal{K}_\varepsilon \subset \mathbb{K}_\varepsilon$. ∎

6 Exponential regularization of energy solutions

In this section, we use the proved regularity of the trajectory attractor $\mathcal{A}_\varepsilon^{tr}$ in order to verify that every energy solution is a sum of an exponentially decaying and a smooth function. This fact, together with the transitivity of exponential attraction, will allow us to establish that the exponential attractor $\mathcal{M}_\varepsilon$, constructed in [25] for the case of *weak* solutions (and small $\varepsilon > 0$) is automatically the exponential attractor in the class of *energy* solutions as well. Moreover, that exponential regularization property will help us to verify the well-posedness and dissipativity of the 2D problem (1.1) in the intermediate (between $\mathcal{V}_0$ and $\mathcal{V}_2$) phase space $\mathcal{V}_1$. To this end, we first formulate the regularity estimate for the solutions on the attractor $\mathcal{A}_\varepsilon^{tr}$ obtained before.

Corollary 6.1. *Let the assumptions of Theorem 5.1 or Theorem 5.3 be satisfied. Then any complete trajectory $U \in \mathcal{K}_\varepsilon$ satisfies*

$$\|u(t)\|_{H^4}^2 + \|u_t(t)\|_{H^2}^2 + \varepsilon \|u_{tt}(t)\|_H^2 \leq C, \tag{6.1}$$

where the constant C is independent of t, ε and of the concrete choice of the trajectory u.

The next result, which gives the analogue of Theorem 4.1 for the forward in time energy solutions, is the main technical tool of this section.

Lemma 6.2. *Let the assumptions of Corollary 6.1 hold, so that the trajectory attractor $\mathcal{A}_\varepsilon^{tr}$ of equation* (1.1) *possesses the regularity* (6.1)*. In addition, suppose that $f \in C^4(\mathbb{R}; \mathbb{R})$. Then, for every energy solution $U \in K_\varepsilon^+$ and every $\sigma > 0$, there exist $T = T(\varepsilon, \kappa, M_u^\varepsilon(0))$ and a function $\tilde{u}(t)$ such that*

$$\|\tilde{u}(t)\|_{H^4}^2 + \|\tilde{u}_t(t)\|_{H^2}^2 + \varepsilon \|\tilde{u}_{tt}(t)\|_H^2 \leq C',\ \ t \geq T, \tag{6.2}$$

where the constant C' depends only on the constant C in (6.1)*. Moreover, this function solves the equation*

$$\varepsilon \tilde{u}_{tt} + \tilde{u}_t + A(A\tilde{u} + f(\tilde{u})) = g + \varphi(t), \tag{6.3}$$

with

$$\|\varphi\|_{C_b([T,+\infty),H)} \leq C\sigma^\kappa, \tag{6.4}$$

where C and $\kappa > 0$ are independent of σ and such that

$$\|u(t) - \tilde{u}(t)\|_H + \|u_t(t) - \tilde{u}_t(t)\|_{H^{-2}} \leq \sigma, \tag{6.5}$$

for all $t \geq T$.

Proof. The proof of this lemma is very similar to the proof of Theorem 4.1. However, instead of the backward attraction to the smooth set of equilibria, we need to use the forward attraction to the smooth trajectory attractor $\mathcal{A}_\varepsilon^{tr}$. Indeed, since the trajectory (u, u_t) is attracted by the trajectory attractor $\mathcal{A}_\varepsilon^{tr}$, then, analogously to (3.15), we have that, for every $\sigma > 0$, there exist $T = T(\varepsilon, \sigma, M_u(0))$ and a trajectory $(u_s, \partial_t u_s) \in \mathcal{K}_\varepsilon$, $s \geq 1$ such that

$$\|u(t) - u_s(t)\|_H + \|\partial_t u(t) - \partial_t u_s(t)\|_{H^{-2}} \leq \sigma, \quad t \in [T + s - 1, T + s + 2] \tag{6.6}$$

for every $s \geq 1$ (compare with (4.6)). Moreover, analogously to Theorem 4.1, due to the regularity (6.1) of the attractor and estimate (6.6), we conclude that

$$\|\partial_t u_s(t) - \partial_t u_{s+1}(t)\|_{H^1} + \|u_s(t) - u_{s+1}(t)\|_{H^3} \leq C\sigma^\kappa, \quad t \in [T+s, T+s+1], \tag{6.7}$$

where the positive constants C and κ are independent of ε, s and the choice of the trajectory u. Finally, defining the function $\tilde{u}(t)$ on the interval $[T+N, T+N+1]$ as follows

$$\tilde{u}(t) := \theta(t-T-N)u_{N+1}(t) + (1-\theta(t-T-N))u_N(t),$$

where the cut-off function θ is the same as in Theorem 4.1 and setting

$$\varphi(t) := \varepsilon\tilde{u}_{tt}(t) + \tilde{u}_t(t) + A(A\tilde{u}(t) + f(\tilde{u}(t))) - g,$$

one can see that the function $\varphi(t)$ satisfies estimate (6.4). Indeed, since u_N and u_{N+1} solve the initial problem (2.11), we have

$$\begin{aligned}\varphi(t) &= 2\varepsilon\theta'\partial_t(u_{N+1} - u_N) + (\varepsilon\theta'' + \theta')(u_{N+1} - u_N) \\ &\quad + A[f(\theta u_{N+1} + (1-\theta)u_N) - \theta f(u_{N+1}) - (1-\theta)f(u_N)],\end{aligned} \tag{6.8}$$

where $\theta = \theta(t-T-N)$. We see that the first two terms are immediately under the control thanks to (6.7) and, in order to estimate the third term, we transform it as follows:

$$\begin{aligned}&f(\theta u_{N+1} + (1-\theta)u_N) - \theta f(u_{N+1}) - (1-\theta)f(u_N) \\ &= \theta(1-\theta)(u_{N+1} - u_N)\int_0^1 [f'(s\tilde{u} + (1-s)u_N) - f'(s\tilde{u} + (1-s)u_{N+1})]\, ds.\end{aligned}$$

Thus, due to (6.7) and the fact that $f \in C^4(\mathbb{R};\mathbb{R})$, the third term in (6.8) is also under control and estimate (6.4) holds. That finishes the proof of the lemma. ∎

The next theorem is analogous to Theorem 4.2, but a bit more delicate since the regularity (6.4) of the function φ given in Lemma 6.2 is slightly lower than in Theorem 4.1.

Theorem 6.3. *Let the assumptions of Lemma 6.2 hold. Then, for any $L > 0$ and any trajectory $U \in K_\varepsilon^+$, there exists a time $T = T(\varepsilon, L, M_u(0))$ such that the equation*

$$\varepsilon v_{tt} + v_t + A(Av + f(v)) + LA^{-1}v = G(t) := g + LA^{-1}u(t) \tag{6.9}$$

has a regular global solution $v(t)$, $t \geq T$ satisfying

$$\varepsilon\|v_{tt}(t)\|_H^2 + \|v_t(t)\|_{H^2}^2 + \|v(t)\|_{H^4}^2 \leq C_L, \tag{6.10}$$

where the constant C_L depends on L, but is independent of ε, t and u. Moreover,

$$\|v_t(t)\|_{L^\infty(\Omega)}^2 + \|v(t)\|_{L^\infty(\Omega)}^2 \leq C, \tag{6.11}$$

where C is independent of L and ε.

PROOF. Step 1. As in Theorem 4.2, in order to solve equation (6.9) we introduce the function $w(t) := v(t) - \tilde{u}(t)$, where $\tilde{u}(t)$ is constructed in Lemma 6.2. Then, this function satisfies

$$\varepsilon w_{tt} + w_t + A(Aw + [f(\tilde{u}(t) + w) - f(\tilde{u}(t))]) + LA^{-1}w = \tilde{G}(t) := LA^{-1}(u(t) - \tilde{u}(t)) - \varphi(t). \tag{6.12}$$

Moreover, due to Lemma 6.2, for any L and any $0 < \sigma < 1/L^2$, we may find $T = T(\varepsilon, L, \sigma, M_u(0))$ such that

$$\|\tilde{G}\|_{C_b([T,+\infty),H)} \leq C\sigma^\kappa, \tag{6.13}$$

where the positive constants C and κ are independent of L, t and u. Then, for small σ, equation (6.12) endowed by the initial data

$$w(T) = 0, \quad w_t(T) = LA^{-1}(u(T) - \tilde{u}(T)), \tag{6.14}$$

can be uniquely solved using the inverse function theorem (analogously to Theorem 4.2). However, since we now do not have the control of the time derivative of φ, we are unable to construct the $\mathcal{V}_3^\varepsilon$-solutions in such way and should restrict ourselves to consider the $\mathcal{V}_1^\varepsilon$-solutions only. Namely, it is not difficult to show using the inverse function theorem that, for sufficiently small σ, there exists a unique global solution $w(t)$ of problem (6.12) with the above initial data such that

$$\varepsilon\|w_t(t)\|_H^2 + \|w(t)\|_{H^2}^2 \le C'\sigma^{2\kappa}, \quad t \ge T, \tag{6.15}$$

where the constant C' is independent of L, t and σ (since very similar arguments have been considered in detail in the proof of Theorem 4.2, we omit the details here).

In addition, from equation (6.12), we see that

$$\varepsilon A^{-1}w_{tt} + A^{-1}w_t = h_w(t) := -Aw(t) - [f(\tilde u(t) + w(t)) - f(\tilde u(t))] + A^{-1}\tilde G(t). \tag{6.16}$$

Thus, due to estimates (6.13) and (6.15), we have

$$\|h_w(t)\|_H \le C\sigma^\kappa, \quad t \ge T.$$

Solving explicitly (6.16) as an ODE with respect to w_t and using the last estimate together with (6.5) and (6.14), we arrive at

$$\|w_t(t)\|_{H^{-2}} \le C\sigma^\kappa, \quad t \ge T, \tag{6.17}$$

where the positive constants C and κ are independent of L, ε, t and u.

Step 2. As we have already mentioned, the regularity (6.15) and (6.17) for the auxiliary problem (6.9) is not sufficient for our purposes and we need to improve it. To this end, we remind that, by the construction of $\tilde u$, we may assume without loss of generality that $\varphi(T) = 0$ and, consequently, initial conditions (6.14) imply that $w_{tt}(T) = 0$. Therefore,

$$v(T) = \tilde u(T), \quad v_t(T) = \tilde u_t(T) + LA^{-1}(u(T) - \tilde u(T)), \quad v_{tt}(T) = \tilde u_{tt}(T). \tag{6.18}$$

Thus, the initial data for the solution v at $t = T$ is more regular, namely (cf. (6.2) and (6.5)),

$$\|v(T)\|_{H^4}^2 + \|v_t(T)\|_{H^2}^2 + \varepsilon\|v_{tt}(T)\|_H^2 \le C, \tag{6.19}$$

where the constant C is independent of ε and L (recall that $\sigma \in (0, 1/L^2)$). Since the (global in time and independent of ε) control of the H^2-norm of the solution $v(t)$ is already obtained (see estimates (6.2) and (6.15)), then $f(v(t))$ is under control. Thus the regularity (6.19) of the initial data implies in a standard way (see e.g., [25] and the proof of Theorem 4.2), that the solution $v(t)$ is indeed more regular and, in particular, estimate (6.10) holds with the constant C_L depending on L, but independent of ε and t.

Step 3. We only need to obtain the estimate (6.11) for the L^∞-norms of v and v_t with the constant C *independent* of L. Moreover, the desired estimate for v is already available (see (6.15) and (6.2)), so we only need to estimate the L^∞-norm of the time derivative. To this end, we use the following standard interpolation inequality together with (6.2) and (6.17)

$$\begin{aligned} &\|v_t(t)\|_{L^\infty} \le \|\tilde u_t(t)\|_{L^\infty} + \|w_t(t)\|_{L^\infty} \\ &\le C + \|w_t(t)\|_{H^{-2}}^{1/8}\|w_t(t)\|_{H^2}^{7/8} \le C + C\sigma^{\kappa/8}(1 + \|v_t(t)\|_{H^2} + \|\tilde u_t(t)\|_{H^2}) \le C + C_L\sigma^{\kappa/8}, \end{aligned} \tag{6.20}$$

where only the constant C_L may depend on L and all of the constants are independent of ε and σ. It only remains to note that, for every fixed L, we may fix $\sigma = \sigma(L)$ in a such way that $C_L\sigma^{\kappa/8} \le 1$. Estimate (6.11) is then an immediate corollary of (6.20) and the theorem is proved. ∎

We are now ready to state and prove the main result of this section.

Theorem 6.4. *Let the assumptions of Lemma 6.2 hold, so that the trajectory attractor $\mathcal{A}_\varepsilon^{tr}$ of equation* (1.1) *satisfies the regularity estimate* (6.1) *with the constant C independent of $\varepsilon \to 0$. Then, there exist a R_0-ball*

$$\mathcal{B} := \{(a,b) \in H^4 \times H^2, \quad \|a\|_{H^4}^2 + \|b\|_{H^2}^2 \le R_0^2\}, \tag{6.21}$$

where the radius R_0 is independent of $\varepsilon \to 0$, and a monotone function Q_ε (depending on ε) such that, for every weak energy solution $U(t) = (u(t), u_t(t)) \in K_\varepsilon^+$, the following estimate holds:

$$\operatorname{dist}_{\mathcal{X}_0^\varepsilon}(U(t), \mathcal{B}) \le Q_\varepsilon(M_u^\varepsilon(0))e^{-\beta t}, \tag{6.22}$$

where the positive exponent β is independent of t and $\varepsilon \to 0$.

Proof. Let $U \in K_\varepsilon^+$ be arbitrary. Then, due to Theorem 6.3, for every $L > 0$, there exists $T = T(\varepsilon, L, M_u^\varepsilon(0))$ (which is independent of the concrete choice of the trajectory U) such that the auxiliary equation (6.9) is globally solvable for $t \ge T$ in the class of regular solutions and the solution $v(t)$ satisfies estimates (6.10) and (6.11). Moreover, crucial for our method is the fact that the constant C in (6.11) is independent of L, ε, T and U. Let now $w(t) := u(t) - v(t)$. Then, this function solves

$$\varepsilon w_{tt} + w_t - A(Aw + [f(v+w) - f(v)]) + LA^{-1}w = 0. \tag{6.23}$$

Recall that we deal only with energy solutions which *can be obtained by Galerkin approximations*. So, we now need to derive the proper estimate for the function w using the Galerkin approximations exactly as in Theorem 4.3. However, in order to avoid the technicalities, we will proceed by formal multiplication of the equation (6.23) by $A^{-1}(w_t + \alpha w)$ (see the proof of Theorem 4.3 for the justification). Then, after integration in space, we arrive at

$$\frac{d}{dt}E_w + \alpha E_w = H_w, \qquad \text{in } [T, +\infty), \tag{6.24}$$

where

$$\begin{aligned} E_w &= \varepsilon\|A^{-1/2}w_t\|_H^2 + \|A^{1/2}w\|_H^2 + L\|A^{-1}w\|_H^2 + 2\alpha\varepsilon(w, w_t) \\ &\quad + \alpha\|A^{-1/2}w\|_H^2 + 2\left(F(v+w) - F(v) - f(v)w, 1\right) \end{aligned} \tag{6.25}$$

and

$$\begin{aligned} H_w &:= -(2 - 3\varepsilon)\|A^{-1/2}w_t\|_H^2 - \alpha\|A^{1/2}w\|_H^2 - \alpha L\|A^{-1}w\|_H^2 \\ &\quad + 2\alpha\left(F(v+w) - F(v) - f(v)w - (f(v+w) - f(v))w, 1\right) \\ &\quad + 2\alpha^2\varepsilon(w, w_t) + \alpha^2\|w\|_H^2 \\ &\quad + 2\left(f(v+w) - f(v) - f'(v)w, v_t\right). \end{aligned} \tag{6.26}$$

Since the constant in estimate (6.11) for the L^∞-norm of v and v_t is independent of L, arguing as in estimates (4.70), we may fix L in such way that (recall also [37, (2.51)])

$$C \ge E_w \ge \gamma\|(w, w_t)\|_{\mathcal{X}_0^\varepsilon}^2, \qquad H_w \le 0, \tag{6.27}$$

where the positive constants C and γ are independent of the concrete choice of the trajectory U.

The Gronwall inequality applied to (6.24) now gives

$$\|(w(t), w_t(t))\|_{\mathcal{X}_0^\varepsilon}^2 \le C([M_u^\varepsilon(T)]^2 + 1)e^{-\alpha(t-T)}, \quad t \ge T.$$

It only remains to recall that $T = T(\varepsilon, M_u^\varepsilon(0))$ (and L is now fixed) and use (3.9) in order to estimate $M_u^\varepsilon(T)$ through $M_u^\varepsilon(0)$. That yields

$$\|(w(t), w_t(t))\|_{\mathcal{X}_0^\varepsilon}^2 \le Q_\varepsilon(M_u^\varepsilon(0))e^{-\alpha t},$$

which implies (6.22) and finishes the proof of the theorem. ∎

Remark 6.5. Although all estimates of auxiliary solutions in the proof given above are uniform with respect to $\varepsilon \to 0$, the monotone function Q_ε in the main estimate (6.22) *depends* on ε since our construction depends crucially on the attraction property to the trajectory attractor $\mathcal{A}_\varepsilon^{tr}$ in a weaker topology (see Corollary 3.5) and the rate of that convergence may be not uniform with respect to ε.

In a fact, it can be non-uniform even with respect to $\varepsilon \in [\varepsilon_1, \varepsilon_2]$ with $\varepsilon_1 > 0$. This problem may be solved in a standard way if we consider the extended trajectory semigroup

$$K^+_{[\varepsilon_1,\varepsilon_2]} := \{(U,\varepsilon),\ U \in K^+_\varepsilon,\ \varepsilon \in [\varepsilon_1,\varepsilon_2]\}, \qquad \mathbb{T}_\ell(U,\varepsilon) := (\mathbb{T}_\ell U, \varepsilon),$$

construct its attractor and use the rate of convergence to the attractor for that semigroup. The difference is that we are now able to approximate the trajectory $U \in K^+_\varepsilon$ not only by the elements of $\mathcal{A}^{tr}_\varepsilon$ but also by the elements of $\mathcal{A}^{tr}_{\varepsilon_n}$ with $\varepsilon_n \to \varepsilon$, which would be enough to obtain the uniformity with respect to ε (see [4] for details). In order to avoid technicalities, we prefer not to give the proof of that uniformity here. However, there is one more essential drawback in the above scheme which cannot be corrected in such an easy way, namely, as we have already mentioned, the function Q_ε in estimate (6.22) depends in a crucial way on the rate of attraction to the attractor and this rate of convergence cannot be found explicitly or expressed in terms of the physical parameters of the system considered (the usual drawback of the global attractors theory), thus, we are factually unable to give any expression for the function Q_ε following the above described arguments. For this reason, we give below an alternative explicit construction of the auxiliary "almost solution" $\tilde{u}(t)$ for the case when $\varepsilon > 0$ is very small. In addition, this construction has another advantage, namely, it is based only on the perturbation arguments and does not use the global Lyapunov functional. This allows to apply it also to non-autonomous cases or to the case of unbounded domains where the Lyapunov functional does not exist. We will return to such issues in more details elsewhere.

The following lemma is the uniform (with respect to $\varepsilon \to 0$) analogue of the key Lemma 6.2.

Lemma 6.6. *Let the assumptions of Lemma 6.2 hold. Then, for every $\sigma > 0$, there exists $\varepsilon_0 = \varepsilon_0(\sigma) > 0$ such that, for any energy solution $U \in K^+_\varepsilon$ with $\varepsilon \le \varepsilon_0$, there exist $T = T(\kappa, M^\varepsilon_u(0))$ and a function $\tilde{u}(t)$ such that*

$$\|\tilde{u}(t)\|^2_{H^4} + \|\tilde{u}_t(t)\|^2_{H^2} + \varepsilon\|\tilde{u}_{tt}(t)\|^2_H \le C, \quad t \ge T, \tag{6.28}$$

where the constant C is independent of u and ε . Moreover, this function solves the equation

$$\varepsilon\tilde{u}_{tt} + \tilde{u}_t + A(A\tilde{u} + f(\tilde{u})) = g + \varphi(t), \tag{6.29}$$

with

$$\|\varphi\|_{C_b([T,+\infty),H)} \le C\sigma^\kappa, \tag{6.30}$$

where C and $\kappa > 0$ are independent of σ and ε and such that

$$\|u(t) - \tilde{u}(t)\|_H + \|u_t(t) - \tilde{u}_t(t)\|_{H^{-2}} \le \sigma, \tag{6.31}$$

for all $t \ge T$. In addition, all of the constants can be expressed explicitly in terms of the physical parameters.

Proof. We define the trajectory $\tilde{u}$ as a solution of the modified Cahn-Hilliard equation

$$\tilde{u}_t + A(A\tilde{u} + f(\tilde{u})) + L_0A^{-1}\tilde{u} = g + L_0A^{-1}u(t), \quad \tilde{u}(0) = u(0), \tag{6.32}$$

where $u \in K^+_\varepsilon$ and $L_0 = L_0(f)$ is a sufficiently large number depending only on f. Then, on the one hand, the difference $u(t) - \tilde{u}(t)$ satisfies the following estimate:

$$\|u(t) - \tilde{u}(t)\|^2_{H^{-1}} \le C\varepsilon\left(1 + Q\left(M^\varepsilon_u(0)\right)e^{-\alpha t}\right), \tag{6.33}$$

where the positive constants C and α and the monotone function Q are independent of u, ε and t (see [25, Prop. 3.4] for the details). Moreover, since we have the uniform control of the H^1-norms of u and $\tilde{u}$, this estimate together with the interpolation inequality gives

$$\|u(t) - \tilde{u}(t)\|^2_H \le C\varepsilon^{1/2}\left(1 + Q\left(M^\varepsilon_u(0)\right)^2 e^{-\alpha t}\right). \tag{6.34}$$

On the other hand, equation (6.32) is a classical parabolic Cahn-Hilliard equation whose solutions possess the standard parabolic smoothing property. Their regularity is restricted only by

the regularity of the nonlinearity f and the external forces $g + L_0A^{-1}u(t)$. In our case, we have $f \in C^3(\mathbb{R}, \mathbb{R})$ and $u \in W^{1,\infty}(\mathbb{R}_+, H^{-1}(\Omega)) \cap L^\infty(\mathbb{R}_+, H^1(\Omega))$. It is then not difficult to verify that this regularity is enough to establish the following estimate for $\tilde u(t)$:

$$\|\tilde u(t)\|_{H^4} + \|\tilde u_t(t)\|_{H^2} + \sqrt{\varepsilon}\|\tilde u_{tt}(t)\|_{L^2} \le C_* + Q(M_u^\varepsilon(0))\frac{1+t^N}{t^N}e^{-\alpha t}, \tag{6.35}$$

where the positive constants C_*, N and α and the monotone function Q are independent of u, ε and t. Since the derivation of this estimate is standard (although a bit technical), we leave it to the reader (see, e.g., [15, Lemma 2.13] for a similar argument).

Thus, the function $\tilde u(t)$ solves equation (6.29) with

$$\varphi(t) := \varepsilon \tilde u_{tt}(t) + L_0A^{-1}(u(t) - \tilde u(t)),$$

and therefore, thanks to (6.34) and (6.35),

$$\|\varphi(t)\|_H \le C\varepsilon^{1/4}\left(1 + Q(M_u^\varepsilon(0))\frac{1+t^N}{t^N}e^{-\alpha t}\right). \tag{6.36}$$

Finally, we only need to estimate the difference $v(t) := u(t) - \tilde u(t)$ in the proper norm. To this end, taking the difference between (2.11) and (6.29), we derive that

$$\varepsilon v_{tt} + v_t = H(t) := -A^2v(t) - A[f(u(t)) - f(\tilde u(t))] - \varphi(t). \tag{6.37}$$

Moreover, using assumption (2.8) on the nonlinearity f and the fact that the L^{p+4}-norms of u and $\tilde u$ are under control (due to the energy estimate), we conclude from estimate (6.34) that

$$\|f(u(t)) - f(\tilde u(t))\|_{L^1} \le (Q(M_u^\varepsilon(0))e^{-\alpha t} + C_*)\|u(t) - \tilde u(t)\|_H^\kappa \le C\varepsilon^{\kappa/4}\left(1 + Q(M_u^\varepsilon(0))e^{-\alpha t}\right)$$

for some positive κ depending only on p. Thus, thanks to (6.34) and (6.36) and the embedding $H^2(\Omega) \hookrightarrow C^0(\bar\Omega)$,

$$\|A^{-2}H(t)\|_H \le C\varepsilon^{\kappa/4}\left(1 + Q(M_u^\varepsilon(0))\frac{1+t^N}{t^N}e^{-\alpha t}\right).$$

Solving explicitly the ODE (6.37) on the interval $[t/2, t]$ and using the last estimate together with the estimate (6.35), we conclude that

$$\|A^{-2}v_t(t)\|_H \le C\varepsilon^{\kappa/4} + C(\varepsilon^{\kappa/4} + e^{-\frac{t}{2\varepsilon}})\left(1 + Q(M_u^\varepsilon(0))\frac{1+t^N}{t^N}e^{-\alpha t}\right).$$

Finally, keeping in mind that the H^{-1}-norm of v_t is under control (due to the energy estimate), we end up with

$$\|u_t(t) - \tilde u_t(t)\|_{H^{-2}} \le C\varepsilon^{\kappa/12} + C(\varepsilon^{\kappa/12} + e^{-\frac{t}{6\varepsilon}})\left(1 + Q(M_u^\varepsilon(0))\frac{1+t^N}{t^N}e^{-\alpha t}\right). \tag{6.38}$$

Estimates (6.34), (6.35), (6.36) and (6.38) show that, indeed, for every $\kappa > 0$, we may fix $\varepsilon_0 = \varepsilon_0(\kappa)$ and $T = T(M_u^\varepsilon(0))$ such that all estimates stated in Lemma 6.6 will be satisfied uniformly with respect to $\varepsilon \le \varepsilon_0$ and $U \in K_\varepsilon^+$. That finishes the proof of the lemma. ∎

Corollary 6.7. *Let the assumptions of Lemma 6.2 hold. Then, there exist $\varepsilon_0 > 0$ and $R_0 > 0$ such that, for every $\varepsilon < \varepsilon_0$, the set*

$$\mathcal{B}_\varepsilon := \{(u_0, u_0') \in H^4 \times H^2 : \|u_0\|_{H^4}^2 + \|u_0'\|_{H^2}^2 \le R_0^2\} \tag{6.39}$$

attracts exponentially all energy solutions $U \in K_\varepsilon^+$:

$$\operatorname{dist}_{\mathcal{X}_0^\varepsilon}((u(t), u_t(t)), \mathcal{B}_\varepsilon) \le Q(M_u^\varepsilon(0))e^{-\alpha t}, \tag{6.40}$$

where the positive constant α and monotone function Q are independent of t, $\varepsilon \le \varepsilon_0$ and of the concrete choice of the trajectory $U \in K_+^\varepsilon$.

Indeed, the derivation of estimate (6.40) is analogous to the proof of Theorem 6.4 with the only difference that, instead of the non-uniform approximations of Lemma 6.2, one should use the uniform approximations of Lemma 6.6.

Remark 6.8. In fact, we have proven a bit more than (6.22) or (6.40). Namely, recalling [37, Def. 4.1] (see also (3.7)), let us consider the M-distance to the set $\mathcal{B}_\varepsilon$ defined by

$$\begin{aligned} &\mathrm{dist}_{M_u^\varepsilon}(t,\mathcal{B}_\varepsilon) \\ &:= \inf\Big\{\liminf_{k\to\infty} d_{\mathcal{X}_0^\varepsilon}(U^{n_k}(t),P_{n_k}\mathcal{B}_\varepsilon)\,:\,U=\Theta^+-\lim_{k\to\infty}U^{n_k},\ U(0)=[\mathcal{X}_0^\varepsilon]^w-\lim_{k\to\infty}U^{n_k}(0)\Big\}. \end{aligned} \tag{6.41}$$

Recall that the external infimum is taken over all the sequences $\{U^{n_k}(t)\}_{k\in\mathbb{N}}$ of Faedo-Galerkin approximations which Θ^+-converge to the given solution U. Then, we may improve estimate (6.40) as follows:

$$\mathrm{dist}_{M_u^\varepsilon}(t,\mathcal{B}_\varepsilon)\le Q(M_u^\varepsilon(0))e^{-\alpha t}. \tag{6.42}$$

This slight generalization is however important for applying the arguments based on the transitivity of exponential attraction, see [16] and next section.

7 Exponential attractors for energy solutions

In this concluding section, we discuss the exponential attractors for problem (2.11). We start with the case of small ε (and the 3D case for definiteness). We first recall that, due to Theorem 5.2 (see [25, Thm. 2.7]), equations (2.11) are globally solvable in the class of more regular $\mathcal{V}_1^\varepsilon$-solutions if the initial data is not large enough and ε is small enough. More precisely, the equation generates a dissipative semigroup $S_\varepsilon(t)$ on the set $\mathbb{B}_\varepsilon\subset\mathcal{V}_1^\varepsilon$ defined by (5.8) and, in particular, the phase space $\mathbb{B}_\varepsilon$ of that semigroup contains an $R(\varepsilon)$-ball of the space $\mathcal{V}_1^\varepsilon$ with $R(\varepsilon)\to\infty$ as $\varepsilon\to 0$.

Remark 7.1. It is worth observing that, on account of the results obtained in [25], we can argue as in [16] to prove that the family $\{(\mathbb{B}_\varepsilon,S_\varepsilon(t))\}_{\varepsilon\in[0,\varepsilon']}$, for some $\varepsilon'>0$ small enough, possesses a uniform family of exponential attractors $\mathcal{M}_\varepsilon\subset\mathcal{V}_3^\varepsilon\cap(H^4\times H)$ with the following properties:

(i) The sets $\mathcal{M}_\varepsilon$ are uniformly bounded in $H^4\times H$ as $\varepsilon\to 0$.

(ii) The sets $\mathcal{M}_\varepsilon$ are compact in $H^3\times H^{-1}$ and their fractal dimensions are uniformly bounded, i.e.,

$$\dim_f(\mathcal{M}_\varepsilon,H^3\times H^{-1})\le C,$$

where C is independent of $\varepsilon\to 0$.

(iii) The uniform exponential attraction property holds

$$\mathrm{dist}_{\mathcal{V}_1^\varepsilon}(S_\varepsilon(t)\mathbb{B}_\varepsilon,\mathcal{M}_\varepsilon)\le Ce^{-\alpha t} \tag{7.1}$$

with positive C and α independent of $\varepsilon\to 0$.

(iv) $\mathcal{M}_\varepsilon$ tends to the limit exponential attractor $\mathcal{M}_0$ as $\varepsilon\to 0$ in the following sense

$$\mathrm{dist}^{symm}_{H^3\times H^{-1}}(\mathcal{M}_\varepsilon,\mathcal{M}_0)\le C\varepsilon^\kappa$$

for some positive κ and C which are independent of ε. We also recall that, as usual, in order to compare the solutions of the hyperbolic equation (2.11) with $\varepsilon>0$ and the solutions of the limit parabolic problem which corresponds to $\varepsilon=0$, one needs to extend the limit parabolic semigroup to the surface (see [4, 16] for details)

$$\mathbb{S}:=\{(u,v)\in H^4\times H,\ \ v=-A(Au+f(u))+g\}.$$

The aim of this section is to verify that the above exponential attractors $\mathcal{M}_\varepsilon$ attract exponentially not only the weak solutions (see (7.1)), but also all energy solutions of problem (2.11). Namely, the following theorem holds.

Theorem 7.2. *Let the assumptions of Theorem 2.2 hold. Then, there exists $\varepsilon_0 > 0$ such that, for every $\varepsilon < \varepsilon_0$, there exists a family of exponential attractors $\mathcal{M}_\varepsilon$ satisfying the properties 1)-4) formulated above and, in addition, for every energy solution $U \in K_\varepsilon^+$ of problem* (2.11)*,*

$$\operatorname{dist}_{\mathcal{X}_0^\varepsilon}((u, u_t), \mathcal{M}_\varepsilon) \le Q(M_u^\varepsilon(0))e^{-\alpha t}, \tag{7.2}$$

where the positive constant α and monotone function Q are independent of ε, t and u.

Proof. Indeed, according to Corollary 6.7 and Remark 6.8,

$$\operatorname{dist}_{M_u^\varepsilon}(t, \mathcal{B}_\varepsilon) \le Q(M_u^\varepsilon(0))e^{-\alpha t}, \tag{7.3}$$

where Q and α are independent of $\varepsilon \le \varepsilon_0$, t and u. On the other hand, thanks to (7.1),

$$\operatorname{dist}_{\mathcal{V}_1^\varepsilon}(S_\varepsilon(t)\mathcal{B}_\varepsilon, \mathcal{M}_\varepsilon) \le Ce^{-\alpha t} \tag{7.4}$$

if $\varepsilon > 0$ is small enough (so that $\mathcal{B}_\varepsilon \subseteq \mathbb{B}_\varepsilon$, cf. (5.8)). Then, keeping in mind that $\mathcal{V}_1^\varepsilon \subset \mathcal{X}_0^\varepsilon$, we see that, in order to prove estimate (7.2) through the transitivity of the exponential attraction (see [16, Thm. 5.1], cf. also [37, Sec. 4]), we only need to check the following version of Lipschitz continuity (see [16, (5.1)]):

$$\operatorname{dist}_{M_u^\varepsilon}(t, V(t)) \le \operatorname{dist}_{M_u^\varepsilon}(0, V(0))e^{Kt}, \tag{7.5}$$

where $\varepsilon > 0$ is small enough, $U \in K_\varepsilon^+$, $V(t) := S_\varepsilon(t)V_0$, $V_0 \in \mathcal{B}_\varepsilon$, is an arbitrary strong solution of equation (2.11) starting from the set $\mathcal{B}_\varepsilon$ and the positive constant K is independent of ε, V and t. This Lipschitz continuity property can be easily verified arguing as in the proof of Theorem 4.4. Indeed, since (due to Theorem 5.2) the solution $V(t) = (v(t), v_t(t))$ exists globally and satisfies the dissipative estimate, one can show that

$$\|v(t)\|_{H^4}^2 + \|v_t(t)\|_{H^2}^2 + \varepsilon\|v_{tt}(t)\|_H^2 \le R_1 = Q(R_0),$$

where the constant R_1 is independent of ε, $V(0) \in \mathcal{B}_\varepsilon$ and t. Therefore, we have the uniform control

$$\|v(t)\|_{L^\infty(\Omega)} + \|v_t(t)\|_{L^\infty(\Omega)} \le C, \tag{7.6}$$

where C is independent of ε, t and $V(0)$. Thus, defining $V^{n_k}(t) = P_{n_k}V(t)$, $z^{n_k} := u^{n_k} - v^{n_k}$, where $u^{n_k}(t)$ are the Faedo-Galerkin approximations to the solution $U \in K_\varepsilon^+$ (see Section 3) and arguing exactly as in the proof of Theorem 4.4, we derive that

$$\frac{d}{dt}E_{z^{n_k}}(t) + \alpha E_{z^{n_k}}(t) \le C\|h^{n_k}(t)\|_{H^{-1}}^2, \tag{7.7}$$

where $E_{z^{n_k}}$ and h^{n_k} are defined as in (4.68) and (4.78) (with $\tilde{u}$ replaced by v) respectively. On the other hand, we have

$$\|h^{n_k}(t)\|_{H^{-1}}^2 \le C\|P_{n_k}(f(v(t)) - f(v^{n_k}(t))\|_{H^1}^2 + CL^2E_{z^{n_k}}(t),$$

and applying the Gronwall inequality to (7.7), we have

$$E_{z^{n_k}}(t) \le E_{z^{n_k}}(0)e^{Kt} + C\int_0^t e^{K(t-s)}\|P_{n_k}(f(v(s)) - f(v^{n_k}(s)))\|_{H^1}\,ds,$$

for some positive C and K independent of u, v, n_k, ε and t. Passing now to the limit $k \to \infty$ and using that the right-hand side tends to zero (note that v is smooth), we derive the desired Lipschitz continuity (7.5). Estimate (7.2) is now a standard corollary of transitivity of exponential attraction. This finishes the proof of the theorem. ∎

We now consider the 2D case with the growth restriction (5.1) for the nonlinearity f (at most cubic growth rate). Then, on account of [26, Thms. 2.2, 3.1 and 5.1]), the 2D problem (2.11) generates a dissipative semigroup $S_\varepsilon(t)$ in the phase space $\mathcal{V}_2^\varepsilon$ for every finite $\varepsilon > 0$ and this semigroup possesses an exponential attractor $\mathcal{M}_\varepsilon$ which is bounded in $\mathcal{V}_3^\varepsilon$. Next theorem shows that this exponential attractor attracts exponentially the energy solutions as well.

Theorem 7.3. *Let the assumptions of Theorem 5.1 hold. Then, for every $\varepsilon > 0$, the exponential attractor $\mathcal{M}_\varepsilon$ for the quasistrong $\mathcal{V}_2^\varepsilon$-solutions constructed in [26] attracts exponentially energy solutions as well. Namely, for any bounded set $B \subset \mathcal{X}_0^\varepsilon = \mathcal{V}_0^\varepsilon$, we have*

$$\operatorname{dist}_{\mathcal{X}_0^\varepsilon}(S_\varepsilon(t)B, \mathcal{M}_\varepsilon) \leq Q(\|B\|_{\mathcal{X}_0^\varepsilon})e^{-\alpha t}, \tag{7.8}$$

where the function Q and constant α are independent of t and B, but may depend on ε. Thus, $\mathcal{M}_\varepsilon$ is an exponential attractor for the solution semigroup $S_\varepsilon(t)$ acting on the energy phase space $\mathcal{X}_0^\varepsilon$ as well.

Indeed, the proof of this theorem repeats word by word the proof of the previous Theorem 7.2, with the only difference that, instead of Corollary 6.7, one should use Theorem 6.4.

To conclude, we apply the proved exponential regularization for the 2D case to one problem which remained unsolved in the previous paper [26]. Namely, we have proved there that problem (2.11) with cubic growth restriction is well posed and dissipative in $\mathcal{V}_0^\varepsilon$ and $\mathcal{V}_2^\varepsilon$, but the dissipativity in the phase space $\mathcal{V}_1^\varepsilon$ occurred surprisingly more delicate and remained an open issue. The next theorem fills this gap without any restriction on ε (compare with [12, Thm. 5.3]).

Theorem 7.4. *Let the assumptions of Theorem 7.3 hold. Then, for every $U(0) \in \mathcal{V}_1^\varepsilon$, problem (2.11) is uniquely solvable in the phase space $\mathcal{V}_1^\varepsilon$ and the following dissipative estimate hold:*

$$\|U(t)\|_{\mathcal{V}_1^\varepsilon} \leq Q(\|U(0)\|_{\mathcal{V}_1^\varepsilon})e^{-\alpha t} + Q(\|g\|_{L^2}), \tag{7.9}$$

where the positive constant α and monotone function Q are independent of t and $U(0)$.

Proof. Obviously, we only need to verify the dissipative estimate (7.9). As usual, we give only the formal derivation which can be easily justified using the Galerkin approximations. The first step is completely standard: we multiply equation (2.11) by $u_t + \gamma u$ where $\gamma > 0$ is a sufficiently small positive number and integrate over x. Then, after the straightforward transformations, we get

$$\frac{d}{dt}Z_u(t) + \kappa Z_u(t) + \kappa\|u_t\|_{L^2}^2 \leq C(1 + \|g\|_{L^2}^2 + \|u(t)\|_{H^1}^2) + \frac{1}{2}|(f''(u)|\nabla u|^2, u_t)|, \tag{7.10}$$

where $\kappa > 0$ is small enough, $C > 0$,

$$Z_u(t) := \frac{1}{2}\left(\varepsilon\|u_t\|_H^2 + \|u\|_{H^2}^2 + (f'(u)\nabla u, \nabla u) + L_0\|u\|_H^2 + 2\gamma\varepsilon(u, u_t)\right),$$

and the constant L_0 is chosen in such way that

$$k\|U(t)\|_{\mathcal{V}_2^\varepsilon}^2 \leq Z_u(t) \leq Q(\|U(t)\|_{\mathcal{V}_2^\varepsilon}) \tag{7.11}$$

for some $k > 0$ and some monotone function Q. Thus, the main problem is how to estimate the last term in the right-hand side of (7.10).

Actually, if we use the Brézis-Gallouet logarithmic inequality together with the fact that $|f''(u)| \leq C(1 + |u|)$ (cf. [8], see also [26, (2.34)]), we obtain

$$\begin{aligned} |(f''(u)|\nabla u|^2, u_t)| &\leq \kappa\|u_t\|_H^2 + C(1 + \|u\|_{L^\infty}^2)\|\nabla u\|_{L^4}^4 \\ &\leq C(1 + \|u\|_{H^1}^2)\ln(1 + \|u\|_{H^2}^2)\|\nabla u\|_{L^4}^4 + \kappa\|u_t\|_H^2 \\ &\leq C'(1 + \|u\|_{H^1}^2)\|\nabla u\|_{L^4}^4 \ln(1 + Z_u(t)) + \kappa\|u_t\|_H^2. \end{aligned} \tag{7.12}$$

Notice now that, if we simply employ

$$\|\nabla u\|_{L^4}^4 \leq C\|\nabla u\|_H^2\|u\|_{H^2}^2 \leq C_1\|u\|_{H^1}^2 Z_u(t),$$

then (using also the $\mathcal{X}_0^\varepsilon$-energy estimate for the solution U) we end up with an inequality of the form

$$\frac{d}{dt}Z_u(t) + \kappa Z_u(t) \leq CZ_u(t)\big(1 + \ln(1 + Z_u(t))\big) + C_1,$$

which is enough to verify the global existence; however, the $\mathcal{V}_2^\varepsilon$-norm of the solution will diverge in time as a double exponential. Thus, in order to obtain the desired dissipative estimate, we have to

proceed more carefully. Namely, thanks to Theorem 6.4, we can split the solution $u(t) = v(t) + w(t)$, where

$$\|v(t)\|_{H^4} \le 2R_0, \quad \|w(t)\|_{H^1} \le Q(\|U(0)\|_{\mathcal{X}_0^\varepsilon})e^{-\alpha t}, \tag{7.13}$$

for some $R_0 > 0$, and estimate the L^4-norm of ∇u as follows

$$\|\nabla u\|_{L^4}^4 \le 8(\|\nabla v\|_{L^4}^4 + \|\nabla w\|_{L^4}^4) \le C(R_0^4 + \|w\|_{H^1}^2 \|w\|_{H^2}^2)$$
$$\le C(1 + Q(\|U(0)\|_{\mathcal{X}_0^\varepsilon})e^{-\alpha t}\|u - v\|_{H^2}^2) \le C + Q(\|U(0)\|_{\mathcal{X}_0^\varepsilon})e^{-\alpha t}(1 + Z_u(t)).$$

Inserting this estimate into the right-hand sides of (7.12) and (7.10) and using the dissipative estimate for the $\mathcal{X}_0^\varepsilon$-energy norm of $U(t)$, we end up with the refined differential inequality

$$\frac{d}{dt}Z_u(t) + \kappa Z_u(t) \le Q(\|U(0)\|_{\mathcal{X}_0^\varepsilon})e^{-\alpha t} Z_u(t)\ln(1 + Z_u(t)) + Q(\|g\|_H) + Q(\|U(0)\|_{\mathcal{X}_0^\varepsilon})e^{-\alpha t}. \tag{7.14}$$

It remains to note that (see proof of [26, Thm. 3.1]) the differential inequality (7.14) gives indeed the desired dissipative estimate (7.9) and finishes the proof of the theorem. ∎

Remark 7.5. As we have already mentioned in Remark 6.5, the functions Q in Theorems 7.3 and 7.4 depend on the rate of convergence of weak energy solutions to the smooth global attractor $\mathcal{A}_\varepsilon$ and, by this reason, we cannot find the explicit expressions of Q in terms of the physical parameters of the system. This drawback can be overcome if we construct the smooth approximate solutions not by using the fact that the energy trajectories tend to the smooth global attractor, but rather observing that these trajectories visit regularly any arbitrarily small neighborhood of the equilibria set (which follows from the existence of a global Lyapunov functional) and the smoothness of the set of equilibria. However, this argument is much more delicate and, in order to avoid the related technicalities, we will not present it here.

References

[1] A. Ambrosetti and G. Prodi, "A Primer of Nonlinear Analysis", Cambridge Studies in Advanced Mathematics, 34, Cambridge University Press, Cambridge, 1995.

[2] A. Babin, *The attractor of a generalized semigroup generated by an elliptic equation in a tube domain,* Russian Acad. Sci. Izv. Math., **44** (1995), 207–223.

[3] A. Babin and M.I. Vishik, *Maximal attractors of semigroups corresponding to evolutionary differential equations,* Mat. Sb., **126(168)** (1985), 397–419.

[4] A.V. Babin and M.I. Vishik, "Attractors of Evolution Equations", North-Holland, Amsterdam, 1992.

[5] J.M. Ball, *Continuity properties and global attractors of generalized semiflows and the Navier-Stokes equations,* J. Nonlinear Sci., **7** (1997), 475–502. Erratum, J. Nonlinear Sci., **8** (1998), 233.

[6] J.M. Ball, *Global attractors for damped semilinear wave equations,* Partial differential equations and applications, Discrete Contin. Dyn. Syst., **10** (2004), 31–52.

[7] A. Bonfoh, M. Grasselli, and A. Miranville, *Singularly perturbed 1D Cahn-Hilliard equation revisited,* submitted.

[8] H. Brézis and T. Gallouet, *Nonlinear Schrödinger evolution equations*, Nonlinear Anal., **4** (1980), 677–681.

[9] J.W. Cahn, *On spinodal decomposition*, Acta Metall., **9** (1961), 795–801.

[10] J.W. Cahn and J.E. Hilliard, *Free energy of a nonuniform system. I. Interfacial free energy*, J. Chem. Phys., **28** (1958), 258–267.

[11] V.V. Chepyzhov and M.I. Vishik, “Attractors for Equations of Mathematical Physics”, American Mathematical Society Colloquium Publications 49, American Mathematical Society, Providence, RI, 2002.

[12] M. Conti and M. Coti Zelati, *Attractors for the non-viscous Cahn-Hilliard with memory in 2D*, Nonlinear Anal., to appear.

[13] M. Conti and G. Mola, *3-D viscous Cahn-Hilliard with memory*, Math. Methods Appl. Sci., **32** (2008), 1370–1395.

[14] A. Debussche, *A singular perturbation of the Cahn-Hilliard equation*, Asymptotic Anal., **4** (1991), 161–185.

[15] M. Efendiev, A. Miranville and S. Zelik, *Exponential attractors for a singular perturbed Cahn-Hilliard system*, Math. Nachr., **272** (2004), 11-31.

[16] P. Fabrie, C. Galusinski, A. Miranville and S. Zelik, *Uniform exponential attractors for a singular perturbed damped wave equation*, Partial differential equations and applications, Discrete Cont. Dyn. Sys., **10** (2004), 211–238.

[17] P. Galenko and D. Jou, *Diffuse-interface model for rapid phase transformations in nonequilibrium systems*, Phys. Rev. E, **71** (2005), 046125 (13 pages).

[18] P. Galenko and V. Lebedev, *Analysis of the dispersion relation in spinodal decomposition of a binary system*, Philos. Mag. Lett., **87** (2007), 821–827.

[19] P. Galenko and V. Lebedev, *Local nonequilibrium effect on spinodal decomposition in a binary system*, Int. J. Thermodyn., **11** (2008), 21–28.

[20] P. Galenko and V. Lebedev, *Nonequilibrium effects in spinodal decomposition of a binary system*, Phys. Lett. A, **372** (2008), 985–989.

[21] S. Gatti, M. Grasselli, A. Miranville, and V. Pata, *On the hyperbolic relaxation of the one-dimensional Cahn-Hilliard equation*, J. Math. Anal. Appl., **312** (2005), 230–247.

[22] S. Gatti, M. Grasselli, A. Miranville, and V. Pata, *Hyperbolic relaxation of the viscous Cahn-Hilliard equation in 3-D*, Math. Models Methods Appl. Sci., **15** (2005), 165–198.

[23] S. Gatti, M. Grasselli, A. Miranville, and V. Pata, *Memory relaxation of the one-dimensional Cahn-Hilliard equation*, Dissipative phase transitions, 101–114, Ser. Adv. Math. Appl. Sci., **71**, World Sci. Publ., Hackensack, NJ, 2006.

[24] C.P. Grant, *Spinodal decomposition for the Cahn-Hilliard equation*, Comm. Partial Differential Equations, **18** (1993), 453–490.

[25] M. Grasselli, G. Schimperna, A. Segatti, and S. Zelik, *On the 3D Cahn-Hilliard equation with inertial term*, J. Evol. Equ., **9** (2009), 371–404.

[26] M. Grasselli, G. Schimperna, and S. Zelik, *On the 2D Cahn-Hilliard equation with inertial term*, Comm. Partial Differential Equations, **34** (2009), 137–170.

[27] M.B. Kania, *Global attractor for the perturbed viscous Cahn-Hilliard equation*, Colloq. Math., **109** (2007), 217–229.

[28] A. Lorenzi and E. Rocca, *Weak solutions for the fully hyperbolic phase-field system of conserved type*, J. Evol. Equ., **7** (2007), 59–78.

[29] S. Maier-Paape and T. Wanner, *Spinodal decomposition for the Cahn-Hilliard equation in higher dimensions: nonlinear dynamics*, Arch. Ration. Mech. Anal., **151** (2000), 187–219.

[30] A. Miranville and S. Zelik, *Attractors for dissipative partial differential equations in bounded and unbounded domains*, Evolutionary equations. Vol. IV, 103–200, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam 2008.

[31] A. Novick-Cohen, *The Cahn-Hilliard equation*, Evolutionary equations. Vol. IV, 201–228, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2008.

[32] R. Rossi, A. Segatti, and U. Stefanelli, *Attractors for gradient flows of non convex functionals and applications*, Arch. Ration. Mech. Anal., **187** (2008), 91–135.

[33] A. Segatti, *On the hyperbolic relaxation of the Cahn-Hilliard equation in 3-D: approximation and long time behaviour*, Math. Models Methods Appl. Sci., **17** (2007), 411–437.

[34] V. Vergara, *A conserved phase field system with memory and relaxed chemical potential*, J. Math. Anal. Appl., **328** (2007), 789–812.

[35] S. Zheng and A.J. Milani, *Global attractors for singular perturbations of the Cahn-Hilliard equations*, J. Differential Equations, **209** (2005), 101–139.

[36] S. Zheng and A.J. Milani, *Exponential attractors and inertial manifolds for singular perturbations of the Cahn-Hilliard equations*, Nonlinear Anal., **57** (2004), 843–877.

[37] S. Zelik, *Asymptotic regularity of singularly perturbed damped wave equations with supercritical nonlinearities*, Discrete Contin. Dyn. Syst., **11** (2004), 351–392.

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